A generalised analogue of the Levi-Civita connection for the reformulation of supergravity as a gravitational theory

Submitted in partial fulfilment of the requirements for the degree of Master of Science of

Imperial College London.

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2021-2022

Acknowledgements

I would like to thank Prof. Waldram for the expert knowledge he shared on generalised geometry, and most of all for his encouraging, uplifting and kind support throughout the writing process.

Contents

1 Introduction

The majority of this work follows the construction of important geometrical structures in the generalised tangent space $\tilde{E} \simeq (\det T^*M)(TM \oplus T^*M)$ described in the paper *Supergravity as Generalised Geometry I: Type II Theories*[1]. This space is an extension of the tangent bundle TM by the cotangent bundle and a real dimension. It serves to describe type II supergravity, to leading order in fermions, as a generalised geometrical analogue to Einstein gravity. The important geometrical structures we will construct here are those that enter in this description of type II supergravity.

In a second part, we will briefly discuss the possibility of finding a generalised geometrical analogue of the Lichnerowicz bound theorem in Einstein gravity. This presents an example of a problem that generalised geometry enables us to pose in a type II supergravity theory, and which could lead to physically significant conclusions.

The fields in Einstein gravity are tensor fields on a differential spacetime manifold, by definition satisfying diffeomorphism invariance (their coordinates are covariant): the fields of this theory are independent of the coordinates we use to describe them. Using the term *structure* to refer to the combination of symmetries on a given space, we can see that Einstein gravity has an internal structure formed of diffeomorphism invariance of its fields on the manifold.

We can express this invariance using such objects as Lie derivatives, where tangent vectors serve as generators of active coordinate changes, and $GL(d, \mathbb{R})$ - covariant connections, where $GL(d, \mathbb{R})$ corresponds to the group of diffeomorphisms. Connections parallel transport a tensor from one tangent space to a neighbouring one; these are solely directional. With a metric on the manifold, we can restrict this connection group to a subgroup G that is compatible with the metric, meaning that a diffeomorphism transformation in this subgroup leaves the metric invariant. This is equivalent to restricting the connections we consider to those with adjoint actions in the Lie algebra of G, and would be giving additional structure to the manifold by imposing that the only neighbouring tangent spaces are those accessible via a G-diffeomorphism. Additional geometrical elements based on these concepts and that we use to describe an Einstein structure are: G - principal frame bundles, composed of tangent space frames, which incorporate the diffeomorphism group structure; the torsion of a connection, which describes how a vector orthogonal to the direction of the connection is parallel transported by the connection, or equivalently how a surface is twisted around the curve defined by a connection; the Riemann curvature tensor of a connection, which measures the difference between the parallel transport of a vector to the same final tangent space but along two different paths, thus measuring the curvature of the space as defined by this connection; the Ricci tensor and Ricci scalar which are formed of the Riemann curvature and metric.

The major perceived obstacle to describing supergravity in the same way, is the coordinate-dependent structure of the supergravity B-field bosonic potential. This field is a local two-form, patched as $B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)}$ on overlapping coordinate patches of the manifold M, $U_i \cap U_j$. The fact that this field is not globally-defined, but patch-dependent, i.e. coordinate dependent, makes its structure non-tensorial, and therefore not incorporated in the tangent space. Due to the particular patching of this field, it has a local symmetry transformation: $B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}$, where $\lambda_{(i)} = \lambda_{(j)}$ so λ is a globally-defined one-form. This B transformation is generated by a one-form and not a tangent vector, defining yet another difference between B and a tensor.

The generalised tangent space we will construct here overcomes this obstacle. By extending the tangent space by the cotangent space with a particular one-form patching, we enable diffeomorphisms generated both by tangent vectors v and cotangent vectors λ, and can replace local diffeomorphism invariance by a larger symmetry group that includes the gauge transformations of

[1]Coimbra, A., Strickland-Constable, C. and Waldram, D. (2011) 'Supergravity as Generalised Geometry I: Type II Theories'. Available at: https://doi.org/10.48550/ARXIV.1107.1733.

the NSNS two-form B. We can further extend this generalised space E to $\tilde{E} \simeq (\det T^*M)E$ without modification of this structure, to include a real dimension for the supergravity dilaton field.

In this paragraph we will list some important results. An essential property of the generalised tangent space is its natural conformal $O(d,d) \times \mathbb{R}^+$ structure, of which the NSNS fields define an $O(p,q) \times O(q,p) = O(9,1) \times O(1,9)$ substructure. We can construct a natural analogue D of the Levi-Civita connection, torsion-free and compatible with the supergravity structure $O(p,q) \times O(q,p) \subset O(d,d)$. This connection is central: We can use D to write the dynamics and symmetries of the supergravity fields in a simple $Spin(9,1) \times Spin(1,9)$ covariant form. However, we have the interesting result that such a connection is not unique. We also arrive, by proceeding in analogy with the Einstein manifold constructions, at an expression for curvature of a generalised connection. However this curvature is not tensorial, unless we restrict it to certain subspaces. Both of these properties may hinder certain geometrical equations and problems we try to extend to generalised geometry, as we will see with the Lichnerowicz bound problem.

In this work, we will focus on the construction of generalised geometry structures and how the NSNS B-field and the symmetry algebra of the NSNS sector are both reflected in the generalised geometry. The structure of this work closely follows that of the paper, as we are following the logical steps in the construction of generalised geometry structures to finally arrive at the expression of the $O(p,q) \times O(q,p)$-covariant generalised connection.

We will start by giving a brief introduction of the fields and structure of type II supergravity, as well as of the geometrical definitions and steps in the construction of the conventional Levi-Civita connection. We will go on to define E, discover an $O(d,d)$ compatible metric, as well as define the generalisations of each ingredient in the construction of the Levi-Civita connection: the frame bundle, tensors, the Lie derivative, connections, torsion and curvature. In a third part we will specify the $O(p,q) \times O(q,p)$ sub-structure compatible with supergravity as well as the general form of a torsion-free, compatible connection, and briefly state the reformulated supergravity equations of motion and symmetry variations. Finally, we will explore the Lichnerowicz bound problem in Einstein gravity, and question its analogue in generalised geometry.

2 Preparation elements

2.1 Type II supergravity

We will look at the structure of $d = 10$ type II supergravity. The paper follows mainly the conventions of the democratic formalism, considers only the leading-order fermionic terms, and rewrites the fermionic sector to better reflect the underlying generalised geometry and later calculations. Here we will introduce the supergravity fields, the bosonic and fermionic pseudo-action and supersymmetry variations, as well as the NSNS bosonic sector. The fields in this sector are the main focus of this paper as these are what we aim and succeed in encoding in the generalised geometry structure. We will merely state in section 4.2 the generalised geometry rewriting of the equations of motion, actions and supersymmetry variations of all the other type II supergravity fields.

2.1.1 Supergravity fields and actions

The fields of $d = 10$ type II supergravity are:

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1\dots\mu_n}, \psi^{+/-}_\mu, \lambda^{+/-}\}$$

where $g_{\mu\nu}$ is the metric, $B_{\mu\nu}$ the 2-form potential, ϕ the dilaton, $A^{(n)}_{\mu_1\dots\mu_n}$ the RR potentials in the democratic formalism[2], $\psi^{+/-}_\mu$ the chiral gravitini, and $\lambda^{+/-}$ the chiral dilatini. $\pm$ distinguish two components with opposite chiralities. The specific nature of these depends on the type of supergravity studies: IIA or IIB.

The bosonic pseudo-action is

$$S_B = \frac{1}{2\kappa}\int \sqrt{-g}[e^{-2\phi}(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2) - \frac{1}{4}\sum \frac{1}{n!}(F^{(B)}_{(n)})^2]$$

where $H = \mathrm{d}B$ and $F^{(B)}_{(n)}$ is the n-form RR field strength, satisfying a self-duality relation that does not follow from varying the action (hence "pseudo" action):

$$F^{(B)}_{(n)} = (-1)^{[n/2]} * F^{(B)}_{(10-n)},$$

where $*$ denotes the Hodge dual.

The fermionic action, keeping only terms that are quadratic in the fermions, can be written as

$$\begin{aligned} S_F = -\frac{1}{2\kappa}\int \sqrt{-g}\Big[&e^{-2\phi}\big(2\bar\psi^{+\mu}\gamma^\nu\nabla_\nu\psi^+_\mu - 4\bar\psi^{+\mu}\nabla_\mu\rho^+ - 2\bar\rho^+\nabla\!\!\!/\rho^+ - \frac{1}{2}\bar\psi^{+\mu}H\!\!\!/\psi^+_\mu \\ &-\bar\psi^+_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^+_\lambda - \frac{1}{2}\rho^+H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi^+_\lambda + \frac{1}{2}\rho^+H\!\!\!/\rho^+\big) \\ &+e^{-2\phi}\Big(2\bar\psi^{-\mu}\gamma^\nu\nabla_\nu\psi^-_\mu - 4\bar\psi^{-\mu}\nabla_\mu\rho^- - 2\bar\rho^-\nabla\!\!\!/\rho^- + \frac{1}{2}\bar\psi^{-\mu}H\!\!\!/\psi^-_\mu + \bar\psi^-_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^-_\lambda - \frac{1}{2}\rho^-H\!\!\!/\rho^-\Big) \\ &-\frac{1}{4}e^{-\phi}\Big(\bar\psi^+_\mu\gamma^\nu F\!\!\!/^{(B)}\gamma^\mu\psi^-_\nu + \rho^+F\!\!\!/^{(B)}\rho^-\Big)\Big], \end{aligned}$$

where ∇ is the Levi-Civita connection, and $\rho^\pm = \gamma^\mu\psi^\pm_\mu - \lambda^\pm$ are the natural combinations that appear in generalised geometry, which we are using instead of $\lambda^\pm$.

The equations of motion for the bosonic fields, setting the fermions to zero, as

$$R_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}e^{2\phi}\sum_n \frac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\dots\lambda_{n-1}}F^{(B)\lambda_1\dots\lambda_{n-1}}_{(\nu} = 0,$$

[2] See paper annex A

$$\nabla^\mu(e^{-2\phi}H_{\mu\nu\lambda}) - \frac{1}{2}\sum_n \frac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} = 0,$$

$$\nabla^2\phi - (\nabla\phi)^2 + \frac{1}{4}R - \frac{1}{48}H^2 = 0,$$

$$\mathrm{d}F^{(B)} - H\wedge F^{(B)} = 0,$$

where

$$F^{(B)} = \sum_n F^{(B)}_{(n)} = \sum_{(n)} e^B\wedge \mathrm{d}A_{n-1},$$

with $e^B = 1 + B + \frac{1}{2}B\wedge B + ...$

The fermionic equations of motion, keeping only terms linear in the fermions (just as we kept only quadratic terms in the action), equate to

$$\gamma^\nu\left[\left(\nabla_\nu \pm \frac{1}{24}H_{\nu\lambda\rho}\gamma^{\lambda\rho} - \partial_\nu\phi\right)\psi^\mp_\mu \mp \frac{1}{2}H_{\nu\mu}{}^\lambda\psi^\mp_\lambda\right] - \left(\nabla_\mu \pm \frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\rho^\mp$$

$$= \frac{1}{16}e^\phi\sum_n(\mp)^{[(n+1)/2}\gamma^\nu \not{F}^{(B)}_{(n)}\gamma_\nu\psi^\pm_\nu,$$

$$\left(\nabla_\mu \pm \frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_\mu\phi\right)\psi^{\mu\mp} - \gamma^\mu\left(\nabla_\mu \pm \frac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi\right)\rho^\mp$$

$$= \frac{1}{16}e^\phi\sum_n(\mp)^{[(n-1)/2]}\not{F}^{(B)}_{(n)}\rho^\pm.$$

2.1.2 Supergravity symmetry variations

- Fermionic and bosonic supersymmetry variations:

  The supersymmetry variations are parametrised by a pair of chiral spinors $\epsilon^\mp$. Keeping only linear terms in the fermionic fields, the supersymmetry variations for the bosons can be written

$$\delta e^a_\mu = \bar\epsilon^+\gamma^a\psi^+_\mu + \bar\epsilon^-\gamma^a\psi^-_\mu,$$

$$\delta B_{\mu\nu} = 2\bar\epsilon^+\gamma_{[\mu}\psi^+_{\nu]} - 2\bar\epsilon^-\gamma_{[\mu}\psi^-_{\nu]},$$

$$\delta\phi - \frac{1}{4}\delta\log(-g) = -\frac{1}{2}\bar\epsilon^+\rho^+ - \frac{1}{2}\bar\epsilon^-\rho^-,$$

$$(e^B\wedge\delta A)^{(n)}_{\mu_1\ldots\mu_n} = \frac{1}{2}(e^{-\phi}\bar\psi^+_\nu\gamma_{\mu_1\ldots\mu_n}\gamma^\nu\epsilon^- - e^{-\phi}\bar\epsilon^+\gamma_{\mu_1\ldots\mu_n}\rho^-)\pm\frac{1}{2}(e^{-\phi}\bar\epsilon^+\gamma^\nu\gamma_{\mu_1\ldots\mu_n}\psi^-_\nu + e^{-\phi}\bar\rho^+\gamma_{\mu_1\ldots\mu_n}\epsilon^-),$$

  where e_μ is a tangent bundle frame that is orthonormal for $g_{\mu\nu}$, and where here the upper sign refers to type IIA, the lower to type IIB.

  For the fermions one has

$$\delta\psi^\mp_\mu = (\nabla_\mu \pm \frac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda})\epsilon^\mp + \frac{1}{16}e^\phi\sum_n(\mp)^{[(n-1)/2]}\not{F}^{(B)}_{(n)}\gamma_\mu\epsilon^\pm,$$

$$\delta\rho^\mp = \gamma^\mu(\nabla_\mu \pm \frac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi)\epsilon^\mp.$$

- NSNS bosonic symmetries and foreshadowing of generalised geometry:

  The structure of the supergravity space is contained in the symmetries of the fields. We will have a look at the symmetries of the NSNS bosonic sector; these will guide the construction of our generalised geometry, which we want to encode the supergravity structure.

  The potential B is locally defined, and given an open cover $\{U_i\}$, is patched across coordinate patches $U_i \cap U_j$ via

$$B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)},$$

  where the one-forms $\Lambda_{(ij)}$ satisfy

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)}$$

  on $U_i \cap U_j \cap U_k$.

  We have the NSNS sector local bosonic gauge symmetry:

$$B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}.$$

  Given the patching of B, $\mathrm{d}\lambda_{(i)}$ must be patched as $\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)}$ on $U_i \cap U_j$. Thus $\omega = d\lambda$ is a 2-form. Recalling the property $d^2 = 0$ of the exterior derivative, we can see that specifying the gauge transformation is equivalent to specifying a closed 2-form $\omega = d\lambda$.

  Infinitesimally, we can summarise the structure, the symmetries, of the NSNS bosonic sector in the following variations of the three fields:

$$\delta_v g = \mathcal{L}_v g, \quad \delta_v \phi = \mathcal{L}_v \phi, \quad \delta_v B_{(i)} \quad \delta_\lambda B_{(i)} = -\mathrm{d}\lambda_{(i)}.$$

  We can combine these diffeomorphism and gauge symmetries by joining the vector and one-form symmetry generators, defining the general variations

$$\delta_{v+\lambda} g = \mathcal{L}_v g, \quad \delta_{v+\lambda} \phi = \mathcal{L}_v \phi, \quad \delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}.$$

  For the symmetry transformations of B to be consistent with its patching, i.e. for them to be intrinsic, independent of the patch (or coordinate system), we have

$$\delta_{v+\lambda(i)} B_{(i)} = \delta_{v+\lambda(j)} B_{(j)} \implies \mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)}.$$

  The Cartan formula gives the equality $\mathcal{L}_X \omega = i_X, d\omega$ for X a vector and ω a differential form. Using this equivalence, we see that by patching $\lambda_{(i)}$ as

$$\lambda_{(i)} = \lambda_{(j)} - i_v d\Lambda_{(ij)},$$

  on $U_i \cap U_j$, we have

$$\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)} - d(i_v d\Lambda_{(ij)}) = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v d\Lambda_{(ij)} + i_v \mathrm{d}^2\Lambda_{(ij)} = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v d\Lambda_{(ij)}$$

  We see that this choice of patching for $\lambda_{(i)}$ gives us the correct patching for its exterior derivative.

  The integration of the above general symmetry variations and patchings into the geometry are the core motivation for of the generalised geometry space, which we will now introduce.

2.2 Differential geometry and the construction of the Levi-Civita connection

Let M be a d-dimensional manifold.

- Vector bundle

  Suppose M is a differential manifold. A manifold E together with a smooth onto map $\pi \to M$ (called the projection) is called a C^r vector bundle of rank k over M if the following three conditions hold:

  - There exists a k-dimensional vector space V such that, for every $p \in M$, $E_p = \pi^{-1}(p)$ is a real vector space isomorphic to V, called the fiber over p;
  - Each point in M is contained in some open set $U \subset M$ such that there is a C^r diffeomorphism
  $$\Phi_U : \pi^{-1}(U) \to U \times V,$$
  with the property that Φ_U restricted to the fiber E_p maps E_p onto $\{p\} \times V$;
  - For any two such open sets U, U' with $U \cap U' \neq 0$, the map
  $$\Phi_U \circ \Phi_{U'}^{-1} : \Phi_{U'}(U \cap U') \to \Phi_U(U \cap U')$$
  is a C^r local vector bundle isomorphism over the identity.

- Dual bundle and its fibers

  $Hom(E, M \times \mathbb{R})$ is the dual bundle to E, its fibers are $Hom(E, M \times \mathbb{R})_p = E_p^*$, the vector space of forms on E_p, isomorphic to E_p. The cotangent bundle T^*M is the dual bundle to TM, its fibers are $T^*M_p = TM_p^*$.

- Dual to a tangent basis

  Writing $\{\hat{e}_a\}$ a basis of the tangent space (tangent bundle fibre) T_xM for $x \in M$, the cotangent basis of T_x^*M that is its dual $\{e^a\}$ satisfies by definition $i_{\hat{e}_a} e^b = \delta_a{}^b$.

- Frame bundle F as a G principal bundle

  The frame bundle F is formed of all bases $\{\hat{e}_a\}$ over M, that is
  $$F = \{(x, \{\hat{e}_a\}) : x \in M, \{\hat{e}_a\} \text{ a basis for } T_xM\}.$$
  On each fibre of F, there is an action of $A^a{}_b \in GL(d, \mathbb{R})$, that brings one basis to another:
  $$\hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b (A^{-1})^b{}_a,$$
  making F a $GL(d, \mathbb{R})$ principal bundle. We say that F has a $GL(d, \mathbb{R})$ structure. This encodes the diffeomorphism structure of E: Given $v \in \Gamma(T_xM)$, we link the above action with the coordinate action
  $$v^a \mapsto v'^a = A^a{}_b v^b,$$
  so that v is unchanged by the total action. This marks the equivalence between a change of basis and a change of coordinates, and gives $GL(d, \mathbb{R})$ the interpretation of the group of possible diffeomorphisms on the tangent space.

- Lie derivative

  The Lie derivative (along a vector) encodes the effect on a tensor of an infinitessimal diffeomorphism. On a general tensor field α and in cooridate indices, one has
  $$\mathcal{L}\alpha^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q} = v^\mu \partial_\mu \alpha^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}$$

$$+(\partial_\mu v^{\mu_1})\alpha^{\mu\mu_2\ldots\mu_p}_{\nu_1\ldots\nu_q} + \ldots + (\partial_\mu v^{\mu_q})\alpha^{\mu\mu_1\ldots\mu_{p-1}\mu}_{\nu_1\ldots\nu_q}$$
$$-(\partial_{\nu_1} v^{\mu})\alpha^{\mu_1\mu_2\ldots\mu_p}_{\mu\nu_2\ldots\nu_q} - \ldots - (\partial_{\nu_q} v^{\mu})\alpha^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\ldots\nu_{q-1}\mu}.$$

One can view the second- and third-line terms as the adjoint action of the $\mathtt{gl}(d,\mathbb{R})$ matrix $a^\mu{}_\nu = \partial_\nu v^\mu$ in the Lie algebra of the structure group, on the tensor field α. We will find an analogous expression for the generalised Lie derivative, with the correct adjoint action. If $\alpha = w$ a vector field, the Lie derivative is equal to the Lie bracket

$$\mathcal{L}_v w = -\mathcal{L}_w v = [v, w].$$

- Connection and its torsion

  A general connection on TM is written in coordinate indices as $\nabla_\mu v^\nu = \partial_\mu v^\nu + \omega_\mu{}^\nu{}_\lambda v^\lambda$.

  The torsion $T \in \Gamma(TM \otimes \Lambda^2 T^*M)$ of ∇ is defined by

  $$T(v,w) = \nabla_v w - \nabla_w v - [v,w]$$

  where $\nabla_v w = v^\mu \nabla_\mu w$, or equivalently

  $$T(v,w) = T^\mu_{\nu\lambda} v^\nu w^\lambda \text{ where } T^\mu_{\nu\lambda} = \omega_\nu{}^\mu{}_\lambda - \omega_\lambda{}^\mu{}_\nu$$

  is the antisymmetrisation of the two lower coordinate indices of ω.

  In a general basis where the connection reads as $\nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b$, the torsion components are written

  $$T^a_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a.$$

  To obtain the natural generalised analogue of the torsion, it is useful to give an equivalent definition in terms of the Lie derivative (which we will see earlier has a natural generalised analogue). Denoting $\mathcal{L}^\nabla_v \alpha$ the analogue of the above Lie derivative with ∂ replaced with ∇, we have

  $$(i_v T)\alpha = \mathcal{L}^\nabla_v \alpha - \mathcal{L}_v \alpha,$$

  where $(i_v T)^\mu{}_\nu = v^\lambda T^\mu_{\lambda\nu}$. From the definition of the connection ∇ and the Lie derivative, we can view $i_v T$ as a section of the $\mathtt{gl}(d,\mathbb{R})$ adjoint bundle, acting on a given tensor field α.

- Curvature, Ricci tensor and Ricci scalar

  The curvature of a connection ∇ is the Riemann tensor $R \in \Gamma(\Lambda^2 T^*M \otimes TM \otimes T^*M)$, defined by

  $$R(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]} w,$$

  which in coordinate indices corresponds to

  $$u^\mu v^\nu R_{\mu\nu}{}^\lambda{}_\rho w^\rho = u^\mu v^\nu([\nabla_\mu n \nabla_\nu] w^\lambda - T^\rho_{\mu\nu} \nabla_\rho w^\lambda).$$

  The Ricci tensor is defined as the trace of the Riemmann curvature

  $$R_{\mu\nu} = R_{\lambda\mu}{}^\lambda{}_\nu.$$

  If the manifold admits a metric g, then we can define the Ricci scalar by

  $$R = g^{\mu\nu} R_{\mu\nu}.$$

- G-structure

  A G-structure is a principal sub-bundle $P \subset F$ with fibre homeomorphic to G.

  As an example, for a given metric g, the $G = O(d)$ sub-bundle is formed by the set of orthonomal bases
$$P = \{(x, \{\hat{e}_a\} \in F : g(\hat{e}_a \hat{e}_b) = \delta_{ab}\},$$
  related by an $O(d) \subset GL(d, \mathbb{R})$ action by definition of the $O(d)$ group.

  At each point of the manifold $x \in M$, the metric is a point in the coset space $GL(d, \mathbb{R})/O(d)$.

  A G-structure can impose topological conditions on the manifold since implies that the tangent space can only be patched (a topological property) using $G \subset GL(d, \mathbb{R})$ transition functions. But there is no such restriction for $O(d)$.

- Compatibility of a connection with a G-structure A connection is compatible with a G-structure $P \subset E$ if the corresponding connection on the principal bundle E reduces to a connection on P, meaning that given a basis $\{\hat{e}_a\}$, one has a set of one-forms $\omega^a{}_b$ in the Lie algebra of G ("taking values in the adjoint representation") given by
$$\nabla_{\partial/\partial x^\mu} \hat{e}_a = \omega_\mu{}^b{}_a \hat{e}_b.$$

  For a metric structure, where the group G of the principal bundle preserves the metric, this is equivalent to the condition $\nabla g = 0$. This stems from an element of the Lie algebra corresponding to an infinitesimal group transformation: $(1 + a)g(1 + a)^T = g + ag + ga^T = g$ for a metric structure, and $ag + ga^T$ corresponds to the adjoint action on both indices of g.

  Furthermore, if there exists a torsion-free compatible connection, the G-structure is said to be torsion-free or (equivalently) integrable to first order. For a metric structure, this compatibility does not imply any further conditions, and the torsion-free, compatible connection, the Levi-Civita connection, is unique.

3 $O(d, d) \times \mathbb{R}^+$ generalised geometry

3.1 Generalised structure bundle $\tilde{E}$

3.1.1 Definition of E

We start by defining the generalised tangent space E as an extension of the tangent space by the cotangent space.

We construct the following exact sequence[3]:
$$0 \xrightarrow{f_1} T^*M \xrightarrow{f_2} E \xrightarrow{f_3} TM \xrightarrow{f_4} 0,$$

To clarify, we can rewrite this, for some atlas (U_α, ϕ_α) on M:

$$(M, 0) = \{(p, \alpha, 0), p \in U_\alpha \in M\} \xrightarrow{f_1 : embedding} \{(p, \alpha, 0 \in (T_{\phi_\alpha(p)}\mathbb{R}^n)^*)\} \subset \{(p, \alpha, \chi \in (T_{\phi_\alpha(p)}\mathbb{R}^n)^*)\}$$
$$\xrightarrow{f_2 : embedding} \{(p, \alpha, 0 \in T_{\phi_\alpha(p)}\mathbb{R}^n + \chi \in (T_{\phi_\alpha(p)}\mathbb{R}^n)^*)\} \subset \{(p, \alpha, \zeta \in T_{\phi_\alpha(p)}\mathbb{R}^n + \chi \in (T_{\phi_\alpha(p)}\mathbb{R}^n)^*)\} = E$$
$$\xrightarrow{f_3 : projection} \{(p, \alpha, \zeta \in T_{\phi_\alpha(p)}\mathbb{R}^n + 0 \in (T_{\phi_\alpha(p)}\mathbb{R}^n)^*)\} \simeq TM \xrightarrow{f_4 : projection} (M, 0).$$

The combination of this sequence and the patching one-forms Λ_{ij} define E and its structure:

[3]Exact sequence: sequence of group homomorphisms where the kernel of each is the image or the previous one: $Ker f_i = Im f_{i-1}$.

For $v_{(i)}$ a section of TU_i and $\lambda_{(i)}$ one of T^*U_i, $V_{(i)} = v_{(i)} + \lambda_{(i)}$ is a section of E over the patch U_i and we set its patching to be

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + (\lambda_{(j)} - i_{v(j)} d\Lambda_{ij}),$$

on $U_i \cap U_j$, where the parentheses separate the covector part of the E section over $U_i \cap U_j$ from its vector part. We see that the v_i globally define a vector: $v_{(i)} = v_{(j)}$, but the $\lambda_{(i)}$ on the other hand do not globally define a one-form. This implies that there is no canonical isomorphism between E and $TM \oplus T^*M$, but we will see later that there does exist an isomorphism between the two.

We note that the structure of the exact sequence, and more specifically the capacity to globally - and continuously[4], by definition of a projection - project from E onto TM, requires the $v_{(i)}$ in a section of E to be globally equivalent to a choice of vector in TM. Then the projection maps a section to a section.

We further note that this patching is consistent with the capacity to embed one-forms into E: If the vector part on each patch of E is null, the E patching of the $\lambda_{(i)}$ is that of a one-form. We cannot however define a projection from E onto T^*M: the cotangent part of an E section that contains a vector part is not globally defined, but is patch-dependent (or coordinate-dependent). There is therefore no way of extracting a section of T^*M from a section of E.

Now that we have defined the generalised tangent space E, we would like to find a metric its patching is consistent with.

3.1.2 A consistent $O(d,d)$ metric

E is consistent with an $O(d,d)$ metric given by, for $V = v + \lambda$, $W = w + \mu$,

$$\langle V, W \rangle = \frac{1}{2}(i_v \mu + i_w \lambda).$$

Before proving this, we recall the definitions of a metric and consistency with a metric:

Definition. *Let M be a topological manifold, $\pi : E \to M$ a vector bundle on M. Then a metric g on E is a bundle map*

$$g : E \times_M E \to M = \{(V, W) \in E \times E : \pi(V) = \pi(W)\} \times \mathbb{R},$$

which is globally defined over M, smooth on each patch, and whose restriction to any fibre over M gives a non-degenerate bilinear form.

Saying that a vector bundle is consistent with a map means that this map corresponds to a metric on E. In this case:

- The bilinearity of the above-defined metric on a fibre E_p stems from the bilinearity of the interior product i;
- The non-degeneracy of the metric is proven by:

$$\left(\langle V, W \rangle = \frac{1}{2}(i_v \mu + i_w \lambda) = 0 \ \forall W = w + \mu \right) \implies \begin{cases} v = 0 \\ \lambda = 0 \end{cases} .$$

[4]Global continuity comes from continuity on each patch combined with being globally well-defined. A continuous map between manifolds by definition maps a continuous section to a continuous section. A section is by definition globally defined, choosing a point in the fibre E_p for every $p \in M$. A continuous section varies differentiably across the fibres.

- Finally and essentially, the metric is globally defined on M, consistent with the patching of E, since the interior product has this property: $i_{v(i)}\lambda_{(i)} = i_{v(j)}\lambda_{(j)}$ on $U_i \cap U_j$.

  Proof. The patching on E gives us $i_{v_{(i)}}\lambda_{(i)} = i_{v_{(j)}}(\lambda_{(j)} - i_{v_{(j)}}\mathrm{d}\Lambda_{(ij)})$. Writing the local tangent frame field as $\{\hat{e}_a\}$ and the dual cotangent frame field $\{e^a\}$, we have $v_{(j)} = v^a_{(j)}\hat{e}_a$ and $\mathrm{d}\Lambda_{(ij)} = \frac{1}{2}(\mathrm{d}\Lambda_{(ij)})_{[ab]}e^a \wedge e^b$.

  $$i_{v_{(i)}}i_{v_{(j)}}\mathrm{d}\Lambda_{(ij)} = v^a_{(j)}v^b_{(i)}\frac{1}{2}(\mathrm{d}\Lambda_{(ij)})_{[ab]} = 0,$$

  since the a, b indices are both symmetrised and anti-symmetrised by different terms of the product. □

We have defined a metric on E, which we will denote η. We will now construct a generalisation of E to be able to include a description of the dilaton, a real field on M which requires a dimension of its own, and transforms as a scalar density.

3.1.3 An extension of E to include the dilaton: $\tilde{E}$

Define $\tilde{E}$ as E weighted by $\det T^*M$:

$$\tilde{E} = \det T^*M \otimes E,$$

where $\det T^*M$ is the space of real fields transforming under the determinant of matrix coordinate transformations.

Given the E metric η, we can now define in terms of bases of E a natural principal bundle with fibre $O(d,d) \times \mathbb{R}^+$, where $O(d,d)$ is defined as the largest group that preserves a specific matrix form for η.

We recall the definition of a principal bundle.

Definition. *A principal G-bundle in the context of smooth manifolds encompasses:*

- *a smooth bundle $\pi : E \to M$, between smooth manifolds E and M,*
- *a Lie group G,*
- *a smooth right action $E \times G \to E$ that preserves the fibres of E,*
- $\forall x \in M, \forall y \in E_x, \begin{cases} G \to E_x \\ g \to yg \end{cases}$ *is a homeomorphism*[5].

Due to the last property of this definition, we will sometimes refer to a fibre of a principal G-bundle as the group G itself.

In the construction of our principal $O(d,d) \times \mathbb{R}^+$-bundle, we first define a conformal frame $\hat{E}_A$ on $\tilde{E}_x$, with $A = 1, ..2d$, as one satisfying

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \Phi^2\eta_{AB} \text{ where } \eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{AB},$$

that is, a basis that is orthonormal[6] up to a frame-dependent conformal factor $\Phi \in \Gamma(\det T^*M)$.

[5] Homeomorphism: Invertible map preserving the topology

[6] The definition of orthonormal here is a frame that satisfies $\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \eta_{AB}$ with the matrix η_{AB} of tha above form.

We can now define the generalised structure bundle $\tilde{F}$ as the coupling of M with all possible conformal bases for $\tilde{E}_x$:

$$\tilde{F} = \{(x, \{\tilde{E}_A\}) \text{ where } x \in M, \text{ and } \{\tilde{E}_A\} \text{ is a conformal basis of } \tilde{E}_x\}.$$

By definition, $O(d,d)$ is the largest group preserving the orthonormal metric.[7] On the other side of the coin, by definition of an orthonormal frame, $O(d,d)$ is the smallest group allowing you to go from one orthonormal frame to all the others in E_x. We can deduce that $O(d,d) \times \mathbb{R}^+$ preserves the conformal metric, and by definition of a conformal frame, $O(d,d) \times \mathbb{R}^+$ can change one conformal frame into any other in a fibre E_x: $\mathbb{R}^+$ allows you to change Φ^2 to any other (positive) Φ^2 in the conformal metric. This can be summarised in the following equivalence:

$$M \in O(d,d) \times \mathbb{R}^+ \iff (M^{-1})^C_A (M^{-1})^D_B \eta_{CD} = \sigma^2 \eta_{AB} \text{ for some positive } \sigma.$$

Furthermore, this group is a Lie matrix group acting on coordinates, so it represents a smooth action on each fibre of $\tilde{F}$.

Thus $\tilde{F}$ is a principal $O(d,d) \times \mathbb{R}^+$-bundle, and we can refer to its fibre as $O(d,d) \times \mathbb{R}^+$.

3.1.4 A natural conformal frame: The coordinate frame

We will recall the meaning of a basis defined by the choice of coordinates on M.

Firstly, a natural local trivialisation for the tangent bundle is the map:

$$\Phi_{U_\alpha} : \begin{cases} \pi^{-1}(U_\alpha) \to U \times \mathbb{R}^{dimM} \\ [r, \alpha, \xi] \to (r, \xi), \xi \in T_{\phi_\alpha(r)} \mathbb{R}^{dimM}, \end{cases}$$

where (U_α, ϕ_α) is an atlas over M. We can deduce a local frame field basis $s_i(r) = \Phi_U^{-1}(r, \hat{e}^\mu = \frac{\partial}{\partial x^\mu})$ for $T_r M$; we will abuse notation and write an element of this basis as $\frac{\partial}{\partial x^\mu}$. Similarly, for the cotangent bundle fibre associated to the same point on M, using the same atlas over M, we can form a fibre basis in the same way, associated to the dual of the above tangent space basis, $\mathrm{d}x^\mu$.

We can now see how this frame plays a role in E. We know that over a given coordinate patch of M there is a canonical isomorphism between $\pi^{-1}(U) \subset E$ and $TU \oplus T^*U$. Therefore at a point $p \in M$, there is a natural basis for E_p given by $\{\hat{E}_A\} = \{\partial/\partial x^\mu\} \cup \{dx^\mu\}$. Given $V \in \Gamma(E)$, over the patch U_i we can write this section in the coordinate frame (locally defined basis) $V = v^\mu (\frac{\partial}{\partial x^\mu}) + \lambda_\mu dx^\mu$, and denote the components of V in this frame by an index M:

$$V^M = \begin{cases} v^\mu & \text{for } M = \mu, \\ \lambda_\mu & \text{for } M = \mu + d \end{cases}.$$

This basis is conformal since we have $i_v \lambda = \lambda_\nu (\mathrm{d}x^\nu \cdot \frac{\partial}{\partial x^\mu}) v^\mu = \lambda_\mu v^\mu$: the η metric corresponds to the trivial $\mathbb{R}^d$ dot product between tangent and cotangent coordinate frame components of two E-vectors.

3.2 Generalised tensors and split frames

3.2.1 Generalised tensors

The $O(d,d)$ metric η defines a vector bundle isomorphism over the identity between E and E^*, meaning that it acts as a linear isomorphism between each fibre E_p and its dual E^*_p. Indeed, for a

[7] The group preserving this metric is in fact isomorphic to the group we would usually call $O(d,d)$, which preserves the diagonal matrix with d ones and d minus ones; this matrix is the diagonalisation of η_{AB}.

given $W \in E$, $f^W : V \mapsto W^A \eta_{AB} V^B$ is an element of the dual bundle of E, E^*. By non-degeneracy of a metric and specifically η, the metric η_x provides an isomorphism $W \mapsto f^W$ between E_x and E^*_x.

The equivalence between E and E^* set by the η vector bundle isomorphism allows us to write sections of E^* as the sections of E that this isomorphism puts them in correspondence with; we therefore write generalised tensors as sections of vector bundle powers of only E and/or $\tilde{E}$. We write such a vector bundle

$$E^{\otimes n}_{(p)} = (\det T^*M)^p \otimes E \otimes ... \otimes E$$

for tensors with n indices, of weight p. By definition this vector bundle is constructed from fibres equal to the vector space products

$$(E^{\otimes n}_{(p)})_x = (\det T^*_x M)^p \otimes E^{\otimes n}_x.$$

Given a group G, we recall that the tensor product of two G-modules V and W is a G-module with

$$\forall a \in G, v \otimes w \in V \otimes W, a(v \otimes w) = av \otimes aw;$$

and given a basis for each fibre, equal to a G-module with $G = O(d,d) \times \mathbb{R}^+$, we have in terms of representations the transformation

$$V^{MN} \to V'^{MN} = \rho^{MN}_{OP} V^{OP} \equiv \rho^M_O \rho^N_P V^{OP},$$

where the tensor representation ρ^{MN}_{OP} is unique up to representation equivalence (corresponding to changes of basis in the different fibres of the vector space product). Thus the tensor vector bundles are constructed from different representations of $O(d,d) \times \mathbb{R}^+$: representations of $O(d,d)$ of definite weight under $\mathbb{R}^+$.

3.2.2 Split frames

We can define an explicit class of conformal frames via a splitting of the generalised tangent space E.

We define a splitting to be a smooth morphism over the identity $f : TM \to E$, that by definition maps a smooth section of TM to a smooth section of E, with the condition that f restricted to T_xM is linear. Choosing the tangent tangent vector portion to remain unchanged, f is therefore entirely defined by $\dim M = d$ local one-forms b^a, $1 \le a \le d$, such that

$$f(\hat{e}_a) = V_a = V_a^B \hat{E}_B = \hat{e}_a + b^a_b e^b, \quad \text{with } V_a \text{ globally defined},$$

where $\{e^a\}$ is the dual basis to $\{\hat{e}_a\}$, and $\{\hat{e}_a\}$ are smooth sections of TM forming a basis on each fibre. These d local one-forms can be replaced by one local two-form defined by $b^a = i_{\hat{e}_a} B \iff b^a_b = B_{ab}$. Recalling the patching on E, the condition that V_a be globally defined is equivalent to:

$$\hat{e}_{a_{(i)}} + i_{\hat{e}_{a(i)}} B_{(i)} = \hat{e}_{a(j)} + i_{\hat{e}_{a(j)}} B_{(j)}$$

$$\iff \begin{cases} \hat{e}_{a(i)} = \hat{e}_{a(j)} \\ i_{\hat{e}_{a(i)}} B_{(i)} = i_{\hat{e}_{a(j)}} B_{(j)} - i_{\hat{e}_{a(j)}} d\Lambda_{(ij)} = i_{\hat{e}_{a(i)}} B_{(j)} - i_{\hat{e}_{a(i)}} d\Lambda_{(ij)} = i_{\hat{e}_{a(i)}} (B_{(j)} - d\Lambda_{(ij)}) \quad \forall\, 1 \le a \le d \end{cases}$$

$$\iff \left\{ B_{(i)} = B_{(j)} - d\Lambda_{(ij)}. \right.$$

Thus defining a splitting is equivalent to specifying a local 2-form B patched as the potential B-field of supergravity!

We can define a split frame $\hat{E}_A$ for $\tilde{E}$ from these globally-defined basis elements in E by complementing them with a set of cotangent basis elements, which we know are globally defined on E, and by adding a smooth common factor to all for the $\mathbb{R}^+$ extension:

$$\hat{E}_A = \begin{cases} \hat{E}_a = (\det e)(\hat{e}_a + i_{\hat{e}_a} B) & for A = a, \\ E^a = (\det e) e^a & for A = a + d, \end{cases}$$

where e is the matrix $e^b{}_\nu = (e^b)_\nu$ of the dual basis elements $\{e^b\}$ in terms of the cotangent coordinate basis elements $\{dx^\mu\}$. A good candidate for a frame conformal factor, $\det e$ transforms correctly under $\det T^*M$, while keeping the same expression: For $e^a \mapsto e'^a = A^a{}_b e^b$, i.e. A a transformation matrix for the cotangent basis, we have

$$\det e^a{}_\nu \mapsto \det \left(A^a{}_b e^b{}_\nu \right) = \begin{cases} \det(A^a{}_b) \det(e^b{}_\nu) = (\det A)(\det e) \\ \det \left((A^a{}_b e^b)_\nu \right) = \det e'^a{}_\nu. \end{cases}$$

To understand why these form a basis on every fibre of $\tilde{E}$ we need only consider that we know $\{\hat{e}_a\} \cup \{e^a\}$ form a basis on the corresponding fibre $E_x \simeq T_x M \oplus T_x^* M$, and $\{i_{\hat{e}_a} B\}$, as local one-forms, are linear combinations of $\{e^a\}$, elements of this set.

We can show that split frames are conformal:

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = (\det e)^2 \eta_{AB}.$$

Indeed,

$$\begin{cases} \left\langle \hat{E}_a, \hat{E}_b \right\rangle = (\det e)^2 i_{\hat{e}_a} i_{\hat{e}_b} B = \frac{1}{2} \left(B_{ab} + B_{ba} \right) = 0 \\ \left\langle \hat{E}^a, \hat{E}^b \right\rangle = 0 \\ \left\langle \hat{E}_a, \hat{E}^b \right\rangle = (\det e)^2 \frac{1}{2} i_{\hat{e}_a} e^b = \frac{1}{2} (\det e)^2 \delta^b_a = \left\langle \hat{E}^b, \hat{E}_a \right\rangle. \end{cases}$$

For $V = v^a \hat{E}_a + \lambda_a E^a \in \Gamma(\tilde{E})$, which we recall we can also write $V = v_{(i)} + \lambda_{(i)}$ on a given patch U_i (where $v_{(i)} \in \Gamma(TU_i)$, $\lambda_{(i)} \in \Gamma(T^* U_i)$), we define the section of $(\det T^*M)(TM \oplus T^*M)$

$$V^{(B)} := v^a (\det e) \hat{e}_a + \lambda_a (\det e) e^a$$

$$= V - v^a_{(i)} (\det e) i_{\hat{e}_{a(i)}} B_{(i)} = V - i_{v(i)} B_{(i)} = v_{(i)} + \lambda_{(i)} - i_{v(i)} B_{(i)}.$$

We can see that the map $V \mapsto V^{(B)}$ defines a smooth vector bundle isomorphism between $\tilde{E}$ and $(\det T^*M)(TM \oplus T^*M)$. Noting that each element of the split frame is globally defined in $\tilde{E}$, $\{\hat{E}_A\}$ are smooth sections of $\tilde{E}$. We then have a bijection between smooth basis sections $\{\hat{E}_A\}$ of $\tilde{E}$ and smooth basis sections $\{(\det e)\hat{e}_a\} \cup \{(\det e)e^a\}$ of $(\det T^*M)(TM \oplus T^*M)$, which gives an isomorphism between the fibres of the two bundles over M, that varies smoothly over M. Indeed, a linear isomorphism between vector spaces is equivalent to a bijection between bases combined with the carrying of coordinates from one basis to the other, which is exactly what this mapping does between two fibres $\tilde{E}_x$ and $(\det T^*M)(TM \oplus T^*M)_x$.

Remark. *We can omit the $\mathbb{R}^+$ extension since the conformal factor is simply carried over in this bundle isomorphism, and say that a B splitting defines a bundle isomorphism $E \simeq TM \oplus T^*M$.*

3.2.3 G_{split}: a frame sub-bundle with all the necessary geometry

The conformal class of split frames defines - or to be more precise, defines a bundle isomorphic to - a sub-bundle of the principal $O(d,d)\times\mathbb{R}^+$ - bundle $\tilde{F}$. These frames are related by coordinate/basis transformations belonging to a group G_{split} of matrices of the form:

$$M = (\det A)^{-1} \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix},$$

where A is invertible and ω is a closed 2-form, i.e. $\mathrm{d}\omega = 0$, that appears here as an antisymmetric matrix: $(\omega)^a{}_b = \omega_{ab}$. G_{split} defines a subgroup $GL(d,\mathbb{R})\times\mathbb{R}^{d(d-1)/2}$ of $O(d,d)\times\mathbb{R}^+$. We can see this isomorphism by identifying the spaces the variables A and ω live in: A is a $d\times d$ real, non-singular matrix, and ω, being a 2-form defined by two anti-symmetric indices each varying from 1 to d, lives in a $d(d-1)/2$-dimensional vector space.

The G_{split} matrix form is defined in split frame indices. Its action on a split frame defined by $\{\hat{e}_a\}$ and B carries out the transformations

$$\begin{cases} \hat{e}_a \to \hat{e}_b (A^{-1})^b{}_a \\ \implies e^a \to A^a{}_b e^b \\ B \to B' = B + \omega, \end{cases}$$

which are exactly the transformations that bring one split frame to another.

Remark. *B' must be patched as B to be a splitting, this requires ω to be a two-form, globally defined. Why it must be closed for B' to be a splitting.. This is more for it to correspond to a bosonic gauge symmetry transformation, no?*

Taking $\omega = 0$, we can show the diffeomorphism part of these transformations:

$$\hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A = \hat{E}_B (\det A) \left[\begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \right]^B{}_A,$$

so we have

$$\hat{E}'_A = \begin{cases} \text{for } A = a \le d: \ \hat{E}'_a = \hat{E}_b (\det A)(A^{-1})^b{}_a = (\det(Ae)) \left(\hat{e}_b (A^{-1})^b{}_a + (A^{-1})^b{}_a i_{\hat{e}_b} B\right) \\ \qquad\qquad\qquad\qquad = (\det e') \left(\hat{e}_b (A^{-1})^b{}_a + i_{\hat{e}_b (A^{-1})^b{}_a} B\right), \\ \text{for } A = d + a: \ E'^a = E^b A^a{}_b = e^b A^a_b \end{cases}$$

where $e' = e'^a_\nu$.

Taking $A = \mathbb{1}$, we can show the effect of ω on B:

$$\hat{E}'_A = \begin{cases} \text{for } A = a \le d: \ \hat{E}_a + E^b (-\omega)^b{}_a = \hat{E}_a - e^b \omega_{ba} = \hat{E}_a + \omega_{ab} e^b = \hat{e}_a + i_{\hat{e}_a}(B + \omega), \\ \text{for } A = a + d: \ E^a. \end{cases}$$

Remark. *Showing these separately does not suffice to prove that any combination of $A \neq \mathbb{1}$ and $\omega \neq 0$ will translate to these transformations; but a less legible calculation proves that this is true.*

Thus the set of split frames defines a G_{split} structure, which from a supergravity point of view, contains all combinations of conventional diffeomorphisms (tangent/cotangent basis/coordinate transformations), and bosonic gauge symmetry transformations of B, $B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}$, where $\omega = \mathrm{d}\lambda$ is a closed one-form. As the patching elements in the definition of $\tilde{E}$ lie entirely in the set of split frames (where the imposed patching of B absorbs the particular patching of E), the structure of $\tilde{E}$ lies in the G_{split} structure.

It will prove useful to define a class of conformal split frames, given by conformal rescaling of the set of split frames by a function ϕ:

$$\hat{E}_A = \begin{cases} \hat{E}_a, = e^{-2\phi}(\det e)\,(\hat{e}_a + i_{\hat{e}_a} B) & \text{for } A = a \leq d \\ E^a = e^{-2\phi}(\det e) e^a & \text{for } A = a + d, \end{cases}$$

defining a $G_{\text{split}} \times \mathbb{R}^+$ sub-bundle of $\tilde{F}$.

Remark. *We know* $(\det e)$ *is already an element of* $\det T^*M$ *as it is real and transforms correctly. However it has no variability, it is entirely determined. The product of* $(\det e)$ *with a positive scalar function transforms the same way and has the added variability that comes with the choice of scalar, spanning all the possible positive* $\det T^*M$ *elements. This puts into relation the group* $\mathbb{R}^+$ *and the space defined by its transformation* $\det T^*M$*,* $\mathbb{R}^+$ *bringing one element of this space to another.*

In analogy with the split case, for $V = V^a \hat{E}_a + \lambda_a E^a = \in \Gamma(\tilde{E})$, we have that

$$V^{(B,\phi)} = e^{2\phi} \left(v_{(i)} + \lambda_{(i)} - i_{v(i)} B_{(i)} \right) = (\det e)\,(v^a \hat{e}_a + \lambda_a e^a) \in (\det T^*M) \otimes (TM \oplus T^*M)$$

is the translation of the components of V in an $\{\hat{E}_A\}$ conformal split frame basis of $\tilde{E}$, onto a $(\det T^*M) \otimes (TM \oplus T^*M)$ frame, relating the split frame components of V to those in the coordinate basis.

3.3 The Dorfman derivative and Courant bracket

3.3.1 The Dorfman derivative or generalised Lie derivative

The generalised tangent space admits a generalisation of the Lie derivative that encodes the bosonic symmetries of the NSNS sector of type II supergravity as well as the diffeomorphism symmetries. Given $V = v + \lambda$, one can define an operator L_V acting on a generalised tensor, which combines an infinitesimal diffeomorphism transformation generated by a tangent vector v and a B-field gauge transformation generated by a one-form λ. We recall that a bosonic gauge symmetry transformation in type II supergravity takes the form: $B'_{(i)} = B_{(i)} - d\lambda_{(i)}$ where $d\lambda_{(i)} = d\lambda_{(j)}$. Thus, the closed w seen in the G_{split} transformation matrix expression, when exact - equal to an exterior derivative, which is indeed closed -, can be interpreted as generating a gauge transformation.

Remark. *We can use infinitesimal diffeomorphisms to characterise movement from one tangent space to a neighbouring tangent space, as moving along* M *is equivalent to changing the coordinates of the point you are looking at. This puts into equivalence the action on a tensor of an infinitesimal diffeomorphism generated by the vector field* v *at that point, and the change of the tensor along the flow defined by* v*, which the conventional Lie derivative aims to describe. This is also equivalent to saying that we describe the diffeomorphism with an active coordinate transformation, instead of passive.*

Here a neighbouring generalised tangent space has not only a new coordinate frame but also a new B*-field. We aim for the generalised Lie derivative to describe the effects of both of these changes on a given tensor's components in the coordinate frame.*

We may also note that the study of the Lie derivative must be done on a given patch, as the coordinate frame is only locally defined. However, the Lie derivative is globally defined.

We recall that conventionally, the action of an infinitesimal diffeomorphism generated by v on a given tensor field is encoded in the Lie derivative acting on all tensor coordinate indices, telling us how the components of the tensor transform under an active coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon V^\mu + O(\epsilon^2)$. Thus by definition the action of the Lie derivative on the the coordinate frame itself (whose components in its own frame are of course constant) is null. The coordinate frame is used as a reference point, the change in the tensor being expressed via the change in its coordinate frame components. We will have the same here, $L_V \hat{E}_N = 0$.

We define the generalised Lie derivative, which we call the Dorfman derivative, as follows: For $V = v + \lambda$ a section of E (which crucially for the Lie derivative implies that v is a global tangent vector field), $W = \omega + \zeta \in E_p$ an E vector of weight p,

$$L_V W = \mathscr{L}_v w + \mathscr{L}_v \zeta - i_w d\lambda.$$

ω and ζ are locally a p-weighted tangent vector field and one-form respectively, so we can write the action of the conventional Lie derivative on these explicitly as:

$$\mathscr{L}_v w^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu,$$

$$\mathscr{L}_v \zeta_\mu = v^\nu \partial_\nu \zeta_\mu + \zeta_\nu \partial_\mu v^\nu + p(\partial_\nu v^\nu)\zeta_\mu.$$

As usual we define the action on a function f to be $L_V f = \mathscr{L}_v f = v^\mu \partial_\mu f$. We can then extend the Dorfman derivative to any tensor using the Leibniz property $L_V(A \otimes B) = L_V A \otimes B + A \otimes L_V B$.

We can obtain the explicit expression for the action of the Dorfman derivative on any tensor by writing its action in a more $O(d,d) \times \mathbb{R}^+$-covariant way, on a generalised coordinate frame index. The part of the action of the generalised Lie derivative that varies the index can then be extended to a generic tensor as the sum of this action on each index.

To find its covariant expression, one first needs to embed the partial derivative operator into the generalised geometry of E:

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu \leq d, \\ 0 & \text{for } M = \mu + d. \end{cases}$$

This is defined as an element of E^*, with the index down. To raise the index and have an element of E we use the metric η^{MN}.

Rewritten in terms of generalised objects, the Lie derivative on an E vector of weight p has the form:

$$L_V W^M = V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N + p\left(\partial_N V^N\right) W^M,$$

where indices are contracted using the $O(d,d)$ metric η_{MN}, and M indexes the coordinate frame $\{\hat{E}_M\}$ component.

Remark. *The $O(d,d)$ metric is constant with respect to the partial derivative ∂_μ, as the coordinate frame is orthonormal; so we can swap the heights of indices in contractions as we please.*

Proof. First, we note

$$\partial^M = \eta^{MN}\partial_N = \begin{cases} 0 & \text{for} M \leq d \\ 2\partial_\mu & \text{for} M = \mu + d \end{cases} \quad \text{where } \eta^{MN} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$W_N = \eta_{NP} W^P = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{NP} W^P = \begin{cases} \frac{1}{2}\zeta_\nu & \text{for } N = \nu \leq d \\ \frac{1}{2}w^\nu & \text{for } N = d + \nu, \end{cases}$$

and

$$i_w \mathrm{d}\lambda = i_w(\partial_\mu \lambda_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = w^\mu \partial_\mu \lambda_\nu \mathrm{d}x^\nu - w^\nu \partial_\mu \lambda_\nu \mathrm{d}x^\mu = w^\nu(\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu)\mathrm{d}x^\mu.$$

Crucially, we have $L_V \hat{E}_N = 0$: Indeed, ?

Thus $L_V W^M = (L_V W)^M$. If can't prove this, maybe just do everything with parentheses

We find the right-hand-side generalised expression $V^N \partial_N W^M + \left(\partial^M V^N\right) W_N - \left(\partial_N V^M\right) W^N + p\left(\partial_N V^N\right) W^M$ to be equal to

$$\begin{cases} v^\nu \partial_\nu w^\mu + 0 - (\partial_\nu v^\mu) w^\nu + p\left(\partial_\nu v^\nu\right) w^\mu = \mathcal{L}_v w^\mu & \text{for } M = \mu \leq d, \\ v^\nu \partial_\nu \zeta_\mu + (2\partial_\mu v^\nu)\frac{1}{2}\zeta_\nu + (2\partial_\mu \lambda_\nu)\frac{1}{2} w^\nu - (\partial_\nu \lambda_\mu) w^\nu + p(\partial_\nu v^\nu)\zeta_\mu = \mathcal{L}_v \zeta_\mu - (i_w \mathrm{d}\lambda)_\mu & \text{for } M = \mu + d. \end{cases}$$

The first case is indeed the tangent vector part $\mathcal{L}_v w$ of $L_V W$, in the tangent coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$, and the second case is the covector part of the generalised Lie derivative, corresponding to $(\mathcal{L}_v \zeta - i_w \mathrm{d}\lambda)$, in the cotangent coordinate basis $\{\mathrm{d}x^\mu\}$.

□

3.3.2 The Dorfman adjoint action

We can rewrite the Dorfman derivative as

$$L_V W^M = V^N \partial_N W^M + \left(\partial^M V_N - \partial_N V^M + \left(\partial_P V^P\right) \delta^M_N\right) W^N,$$

which has the exact same form as the conventional Lie derivative acting on a conventional tensor, but with the adjoint action $-m^M_{\ N} = \partial^M V_N - \partial_N V^M + \left(\partial_P V^P\right) \delta^M_N$ living in the Lie algebra $o(d,d) \oplus \mathbb{R}$ instead of $gl(d,\mathbb{R})$ like the conventional adjoint action $a^\mu_{\ \nu} = \partial_\nu v^\mu$.

Indeed, we can show that m is an element of this Lie algebra, being of the form:

$$m.W = \left[\begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} - p\mathrm{tr}a\mathbb{1}\right] \begin{pmatrix} \omega \\ \zeta \end{pmatrix}$$

where $w_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$, and $a^\mu_{\ \nu} = \partial_\nu v^\mu$ is the conventional adjoint action in the Lie algebra of the non-singular, real $d \times d$ matrices for changes of tangent basis.

We want to verify 1) that a matrix of this form does in fact live in the Lie algebra of $O(d,d) \times \mathbb{R}^+$, and 2) that the action described above is in fact equal to the action of m.

1) For $M \in O(d,d)$, by definition: $M^T \eta M = \eta$. To first order in the coefficients of m defined by $M = \mathbb{1} + m$, we have

$$M^T \eta M = \left(\mathbb{1} + m^T\right) \eta \left(\mathbb{1} + m\right) = \eta + m^T \eta + \eta m$$

So $gl(2d,\mathbb{R}) \supset o(d,d) = \{m^T = -\eta m \eta^{-1}\}$, which is equivalent to saying m is of the form:

$$m^M_{\ N} = \begin{pmatrix} A & B \\ C & -A^{-T} \end{pmatrix} \text{ with } B \text{ and } C \text{ antisymmetric.}$$

These conditions are indeed satisfied by the matrix $\begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix}$ since ω is a 2-form and therefore has antisymmetric coefficients $\omega_{\mu\nu}$, so

$$\begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} - p\mathrm{tr}a\mathbb{1} \in o(d,d) \oplus \mathbb{R}.$$

Remark. *The $o(d,d)$ Lie algebra condition is equivalent to:*

$$m^{MN} = m^M_{\ P} \eta^{PN} = \begin{pmatrix} A & B \\ C & A^{-T} \end{pmatrix} \times 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 \begin{pmatrix} B & A \\ -A^T & C \end{pmatrix}^{MN}, \textit{ i.e. } m^{MN} \textit{ is antisymmetric.}$$

We will use this result in a future section.

In fact we can even be more specific about the adjoint action of the Dorfman derivative: m acts in the Lie algebra of the G_{split} subgroup of G. Indeed, taking the corresponding form of $M \in G_{\text{split}}$, and choosing to write $-\omega$ for the closed 2-form, we have

$$M = (\det A)^{-1} \begin{pmatrix} A & 0 \\ -\omega A & (A^{-1})^T \end{pmatrix} = (1 - \text{tr}a) \begin{pmatrix} \mathbb{1} + a & 0 \\ -\omega(\mathbb{1} + a) & \mathbb{1} - a^T \end{pmatrix} = \mathbb{1} - \text{tr}a\mathbb{1} + \begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix}$$

to first order in $(A - \mathbb{1})^a{}_b = a^a{}_b$ and ω_{ab}. This is exactly the form of the adjoint action m in $L_V W$, where $p = 1$ ($W \in \tilde{E}$). Note that the fact that $M \in G_{\text{split}}$ is written in a split basis, not the coordinate basis like the adjoint action matrix m, is unimportant. A change of basis just marks an isomorphism between equivalent groups or Lie algebras, or equivalent representations of the same group. If we put the set of G_{split} transformation matrices in the coordinate frame, the set of Lie algebra matrices we would find would be isomorphic to the set of adjoint action matrices m.

Remark. *From the form of $M \in G_{split}$, and the derivation of the Lie algebra matrices, we can deduce the form M would need to have for $p > 1$: $(\det A)^{-1}$ would need to be replaced with $(\det A)^{-p}$.*

2) We will now verify that the action of m in the Lie derivative $L_V W^M = V^N \partial_N W^M - m^M_N W^N$ is equal to this adjoint action:

$$\begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} \begin{pmatrix} \omega \\ \zeta \end{pmatrix} - p(\text{tr}a) \begin{pmatrix} \omega \\ \zeta \end{pmatrix} = \begin{pmatrix} aw - p(\text{tr}a)w \\ -\omega w - a^T\zeta - p(\text{tr}a)\zeta \end{pmatrix}$$

$$= \begin{pmatrix} (\partial_\nu v^\mu)w^\nu - p(\partial_\nu v^\nu)w^\mu \\ -(\partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu)w^\nu - (\partial_\mu v^\nu)\zeta_\nu - p(\partial_\nu v^\nu)\zeta_\mu \end{pmatrix} = -\begin{pmatrix} \mathcal{L}_v w^\mu - v^\nu \partial_\nu w^\mu \\ \mathcal{L}_v \zeta_\mu - v^\nu \partial_\nu \zeta_\mu \end{pmatrix}$$

$$= -\left(L_V W^M - V^N \partial_N W^M\right) = \left(m^M_N W^N\right).$$

This Lie derivative can be naturally extended to an arbitrary $O(d,d) \times \mathbb{R}^+$ tensor $\alpha \in \Gamma(E^{\otimes n}_{(p)})$:

$$L_V \alpha^{M_1 \ldots M_n} = V^N \partial_N \alpha^{M_1 \ldots M_n} + (\partial^{M_1} V^N - \partial^N V^{M_1}) \alpha_N^{\ M_2 \ldots M_n} + \ldots + (\partial^{M_n} V^N - \partial^N V^{M_n}) \alpha_N^{\ M_1 \ldots M_{n-1}} + p(\partial_N V^N) W^M.$$

3.3.3 The Courant bracket

The Dorfman derivative by definition must be taken with respect to a section of E ($n = 1$,$p = 0$), i.e. $V \in \Gamma(E)$ in $L_V W$, but if we also take $W \in \Gamma(E)$, one can define the antisymmetrisation of the Dorfman derivative: the Courant bracket. For $W = w + \zeta$ and $V = v + \lambda$, we have

$$[\![V, W]\!] = \frac{1}{2}(L_V W - L_W V)$$

We can show that this is equal to

$$[\![V, W]\!] = [v, w] + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda - \frac{1}{2} d(i_v \zeta - i_w \lambda).$$

Proof.

$$[\![V, W]\!] = \frac{1}{2} \left((\mathcal{L}_v w + \mathcal{L}_v \zeta - i_w d\lambda) - (\mathcal{L}_w v + \mathcal{L}_w \lambda - i_v d\zeta) \right) = \frac{1}{2} \left([v, w] - [w, v] + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda + i_v d\zeta - i_w d\lambda \right),$$

where we identified $\mathcal{L}_v w = [v, w]$. We can use the Cartan formula for a differential form λ, $\mathcal{L}_v \lambda = \{d, i_v\} \lambda$, to write

$$i_v d\zeta - i_w d\lambda = \mathcal{L}_v \zeta - d(i_v \zeta) - \mathcal{L}_w \lambda + d(i_w \lambda),$$

and finally $[\![V, W]\!] = \frac{1}{2} \left(2[v, w] - d(i_v \zeta) + d(i_w \lambda)\right) + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda = [v, w] + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda - \frac{1}{2} d\left(i_v \zeta - i_w \lambda\right)$ where we used the linearity of the exterior derivative d. $\square$

To write the Courant bracket in an $O(d,d)$ covariant form, we antisymmetrise the covariant form of the Dorfman derivative for $p=0$:

$$\frac{1}{2}\left(V^N\partial_N W^M + (\partial^M V^N)W_N - \left(\partial_N V^M\right)W^N - (V \leftrightarrow W)\right)$$

We note that the sum of the first and third terms is already antisymmetric on V and W. All we need to do is antisymmetrise the second term:

$$[\![V,W]\!]^M = V^N\partial_N W^M - W^N\partial_N V^M - \frac{1}{2}\left(V_N\partial^M W^N - W_N\partial^M V^N\right),$$

giving us our covariant Courant bracket expression.

3.4 Generalised $O(d,d)\times\mathbb{R}$ connections and torsion

We want to define generalised connections and torsion to be able to explore the possibility of defining a generalised curvature.

3.4.1 Generalised connections

We are interested in generalised connections compatible with the $O(d,d)\times\mathbb{R}^+$ structure.

Define a 1st-order linear differential operator D such that in frame indices, for $W\in\Gamma(\tilde{E})$,

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B.$$

For D to be a compatible connection with the $G=O(d,d)\times\mathbb{R}^+$ structure, we need $\tilde{\Omega}_M{}^A{}_B$ to live in the Lie algebra of G:

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M\delta^A_B,$$

where Λ is the $\mathbb{R}^+$ part of the connection, and Ω the $O(d,d)$ part satisfying

$$\Omega_M^{AB} = -\Omega_M^{BA},$$

as we previously explained. We have a natural extension of the action of D to any generalised tensor; for $\alpha\in\Gamma(E_{(p)}^{\otimes n})$ we have

$$D_M\alpha^{A_1\ldots A_n} = \partial_M\alpha^{A_1\ldots A_n} + \Omega_M{}^{A_1}{}_B\alpha^{BA_2\ldots A_n} + \ldots + \Omega_M{}^{A_n}{}_B\alpha^{A_1\ldots A_{n-1}B} - p\Lambda_M\alpha^{A_1\ldots A_n}.$$

Given a conventional connection ∇ and a conformal split frame where we recall $\Phi = e^{-2\phi}(\det e)$, we have a corresponding generalised connection which we will denote $D^{\nabla(1)}$.

Writing $W\in\Gamma(\tilde{E})$ in the conformal split frame, we have

$$W = W^A\hat{E}_A = w^a\hat{E}_a + \zeta_a E^a,$$

and by construction of a split frame, $w = w^a(\det e)\hat{e}_a \in \Gamma((\det T^*M)\otimes TM)$ and $\zeta = \zeta_a(\det e)e^a \in \Gamma((\det T^*M)\otimes T^*M)$. So $\nabla_\mu w^a$ and $\nabla_\mu\zeta_a$ are well defined, which we can use to define a generalised connection whose action on $\tilde{E}$ corresponds to these conventional actions on the split frame coordinates: If $M\leq d$ we have

$$(D_M^{\nabla(1)}W^A)\hat{E}_A = \begin{cases}\nabla_\mu w^a & \text{for } A = a\leq d\\ \nabla_\mu\zeta_a & \text{for } A = a+d\end{cases},$$

otherwise $D_M^{\nabla(1)}W^A = 0$. We can say that the conformal split frame lifts the connection ∇ to an action on $\tilde{E}$.

3.4.2 Generalised torsion

In analogy to the conventional definition of the torsion, we define the generalised torsion T of a generalised connection D as, for α a generalised tensor and $L^D_V\alpha$ the Dorfman derivative with D replacing ∂, we define

$$T(V)\cdot\alpha = L^D_V\alpha - L_V\alpha,$$

where $T(V)$ acts via the adjoint representation on α. Indeed, denoting $m^M_{\ P}$ and $m'^M_{\ P}$ the adjoint actions of $L^D_V\alpha$ and $L_V\alpha$ respectively, we have for W a generalised vector (to simplify we reduce the number of indices to one)

$$T(V)W^M = V^N\tilde{\Omega}_N{}^M{}_P W^P + m^M_{\ P}W^P - m'^M_{\ P}W^P,$$

where $\tilde{\Omega}_N{}^M{}_P \forall N$, $m^M_{\ P}$ and $m'^M_{\ P}$ all live in the Lie algebra of G, hence so does $T(V)$.

So

$$T : \Gamma(E) \to \Gamma(\mathrm{ad}\tilde{F})$$

where $\mathrm{ad}\tilde{F}$ represents the $o(d,d)\oplus\mathbb{R}$ adjoint representation bundle associated to $\tilde{F}$: an element of a fibre $(\mathrm{ad}\tilde{F})_x$ is a matrix element of the Lie algebra of G, the group associated to the bundle $\tilde{F}$. It is written in a basis of $\tilde{F}_x$ and acts on coordinates and basis elements of $\tilde{E}_x$.

We have the isomorphism $\mathrm{ad}(\tilde{F}) \simeq \Lambda^2E\oplus\mathbb{R}$. Indeed, we saw that the space $o(d,d)$ was equivalent to the space of antisymmetric matrices with E indices, and therefore to the space of antisymmetric 2-index E tensors.

We can view the torsion T as a tensor $T \in \Gamma(E\otimes \mathrm{ad}(\tilde{F}))$, since it sends E_x to $(\mathrm{ad}(\tilde{F}))_x$ and is linear in $V \in \Gamma(E)$. Indeed, $T(V)$ being an element of the Lie algebra and linear in V we have the indices $T(V)^M_{\ N} = V^P T^M_{\ PN}$; $T^M_{\ PN}$ is corresponds to the tensor $T \in \Gamma(E^*\otimes\mathrm{ad}(\tilde{F})) \simeq \Gamma(E\otimes\mathrm{ad}(\tilde{F}))$. By definition, the action of $T(V)$ on each $\tilde{E}$ index is the same, so it suffices to explicitly express the action of $T(V)$ on a generalised vector $W \in \tilde{E}$ to have its action on any generalised tensor.

Denoting $\{\hat{E}_A\}$ a general conformal basis for $\tilde{E}$ with $\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \Phi^2\eta_{AB}$, $\{\Phi^{-1}\hat{E}_A\}$ is an orthonormal basis for E. We define the different components as follows:

$$\text{for } V\in\Gamma(E),\ V = V^A\Phi^{-1}\hat{E}_A, \ \text{ and for } W\in\Gamma(\tilde{E}),\ W = W^A\hat{E}_A.$$

Given a connection $D_MW^A = \partial_MW^A + \tilde{\Omega}_M{}^A{}_BW^B$, one has

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B\eta_{AC} - \Phi^{-2}\left\langle \hat{E}_A, L_{\Phi^{-1}\hat{E}_B}\hat{E}_C\right\rangle,$$

where indices are lowered with η_{AB}.

Proof. Recall for $W\in\Gamma(\tilde{E})$,

$$D_MW^A = \partial_MW^A + \tilde{\Omega}_M{}^A{}_BW^B,$$
$$L_VW^M = V^N\partial_NW^M + (\partial^MV^N - \partial^NV^M)W_N + (\partial_NV^N)W^M,$$

and by definition of the generalised connection D

$$L^D_V\hat{E}_C = 0\ \forall C.$$

Writing $V\in\Gamma(E)$ with the components $V = V^B(\Phi^{-1}\hat{E}_B)$, and any quantity in $\tilde{E}$ as $W = W^A\hat{E}_A \in \Gamma(\tilde{E})$: e.g. $\left(L_{\Phi^{-1}\hat{E}_B}\right)^C\hat{E}_C$, $[T(V)(W)]^A\,\hat{E}_A$, we have

$$[T(V)W]^A = \left[T(V)(W^C\hat{E}_C)\right]^A = T(V)(W^C)\hat{E}_C^{\ A} + W^C\left[T(V)(\hat{E}_C)\right]^A$$

$$= V^B \tilde{\Omega}_B{}^A{}_C W^C + (\tilde{\Omega}^A_{CB} V^B - \tilde{\Omega}_C{}^A{}_B V^B) W^C + \tilde{\Omega}_D{}^D{}_B V^B (\delta^A_C W^C) - W^C V^B (L_{\Phi^{-1}\hat{E}_B} \hat{E}_C)^A$$

$$= T(V)^A{}_C W^C = V^B T^A_{BC} W^C \text{ by definition of these two objects.}$$

So, lowering the D index of T^D_{BC} with η_{DA}, we have

$$T_{ABC} = \tilde{\Omega}_{BAC} + \tilde{\Omega}_{ACB} - \tilde{\Omega}_{CAB} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - (L_{\Phi^{-1}\hat{E}_B} \hat{E}_C)^D \eta_{DA}.$$

Knowing the 2 last of Ω's indices are already antisymmetric by definition, we have

$$\tilde{\Omega}_{[ABC]} = \Omega_{[ABC]} - \Lambda_{[A}\eta_{BC]} = \Omega_{[ABC]} = \frac{1}{3}(\Omega_{ABC} + \Omega_{BCA} + \Omega_{CAB}).$$

We can also note

$$\left\langle L_{\phi^{-1}\hat{E}_B} \hat{E}_C, \hat{E}_A \right\rangle = (L_{\phi^{-1}\hat{E}_B} \hat{E}_C)^D \left\langle \hat{E}_D, \hat{E}_A \right\rangle = (L_{\phi^{-1}\hat{E}_B} \hat{E}_C)^D \Phi^2 \eta_{DA}.$$

So $(L_{\Phi^{-1}\hat{E}_B} \hat{E}_C)^D \eta_{DA} = \Phi^{-2} \left\langle L_{\phi^{-1}\hat{E}_B} \hat{E}_C, \hat{E}_A \right\rangle$, and we have the final expression

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - \Phi^{-2} \left\langle \hat{E}_A, L_{\Phi^{-1}\hat{E}_B} \hat{E}_C \right\rangle,$$

where $\tilde{\Omega}_{[ABC]} = \Omega_{[ABC]}$. □

Remark. *The middle index B is an E index, the others are $\tilde{E}$ indices.*

One might expect $T \in \Gamma\left((E \otimes \Lambda^2 E) \oplus E\right)$ by distributing the product of E with $\mathrm{ad}\tilde{F} \simeq \Lambda^2 E \oplus \mathbb{R}$, but fewer components actually enter the torsion. We can see this by looking at the above expression of the torsion tensor: the first term is totally antisymmetric and lives in E, since the $\mathbb{R}$ part $\Lambda_M \eta_{AB}$ of $\tilde{\Omega}$ disappears with antisymmetrisation. The second term has one E index, B, and the third term acts as a constant, independent of Ω in the Dorfman derivative. The generalised torsion therefore lives in a space isomorphic to the direct sum $\Gamma(\Lambda^3 E \oplus E)$.

We can see this more directly in the two components of the coordinate basis $\hat{E}_N$, where we have $L_V \hat{E}_N = 0$, therefore

$$T^M_{PN} = (T_1)^M_{PN} - (T_2)_P \delta^M_N,$$

with

$$(T_1)_{MNP} = -3\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]},$$

$$(T_2)_M = -\tilde{\Omega}_Q{}^Q{}_M = \Lambda_M - \Omega_Q{}^Q{}_M.$$

3.4.3 Generalised torsion for D^∇, where ∇ is torsion-free

Here we will calculate the torsion for the generalised connection D^∇ associated to a torsion-free conventional connection ∇. This explicit calculation will facilitate the writing of other generalised connections and torsions; notably it will help us derive the form of a generalised connection that we impose to be torsion-free.

Recall that D^∇ is defined with respect to the components in the split conformal frame $\{\hat{E}_A\}$, with conformal factor $\Phi = e^{-2\phi}(\det e)$.

We are interested in the coordinate frame components T_1 and T_2. To obtain these we will first calculate

1. the torsion T^A_{BC} in the split conformal frame, where we know the expression of D^∇,
2. $\left(L_{\Phi^{-1}\hat{E}_B} \hat{E}_C\right)^A$, (minus) the third term in the expression of T^A_{BC}.

The sum of these is equal to $(T_1)^A{}_{BC} + (T_2)_B \delta^A{}_C$, which we will use to deduce each component separately in the split conformal frame. We can then raise all the indices with η^{AB} to remove all dual indices, and finally take $\hat{e}_a$ in $\hat{E}_A$ to be the coordinate frame $\{\frac{\partial}{\partial x^\mu}\}$, and find the components T_1 and T_2 in the generalised coordinate frame $\{\hat{E}_N\}$.

1. To find $T^A{}_{BC}$, we write for $V \in \Gamma(E)$, $W \in \Gamma(\tilde{E})$,

$$T(V) \cdot W = L^D_V W - L_V W = (\mathcal{L}^\nabla_v w + \mathcal{L}^\nabla_v \zeta - i_w \mathrm{d}^\nabla \lambda) - (\mathcal{L}_v w + \mathcal{L}_v \zeta - i_w \mathrm{d}\lambda),$$

which for a torsion-free conventional connection

$$\begin{cases} \nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b \\ \nabla_\mu \lambda_a = \partial_\mu \lambda_a - \omega_\mu{}^b{}_a \lambda_b \end{cases} \qquad \text{in the frames } \{\hat{e}_a\} \text{ and } \{e^a\},$$

where $\mathcal{L}^\nabla_v w - \mathcal{L}_v w = 0$ and $\mathcal{L}^\nabla_v \zeta - \mathcal{L}_v \zeta = 0$, becomes

$$T(V) \cdot W = i_w(\mathrm{d}\lambda - \mathrm{d}^\nabla \lambda) = i_w(\omega_c{}^b{}_a \lambda_b e^c \wedge e^a) = w^c \lambda_b (\omega_c{}^b{}_a - \omega_a{}^b{}_c) e^a = W^C V^B T^A{}_{BC} \hat{E}_A.$$

2. We have generally

$$L_{\Phi^{-1}\hat{E}_B} \hat{E}_C = (L_{\Phi^{-1}\hat{E}_B} \Phi) \Phi^{-1} \hat{E}_C + \Phi(L_{\Phi^{-1}\hat{E}_B}(\Phi^{-1}\hat{E}_C))$$

where here, in the conformal split frame,

$$L_{\Phi^{-1}\hat{E}_B} \Phi = \begin{cases} -e^{-2\phi}(\det e)(i_{\hat{e}_b} i_{\hat{e}_d} \mathrm{d}e^d + 2 i_{\hat{e}_b} \mathrm{d}\phi) & \text{for } B = b \leq d \\ 0 & \text{for } B = b + d \end{cases}$$

Proof. First, we note that

$$L_{\Phi^{-1}\hat{E}_B} \Phi = \begin{cases} \mathcal{L}_{\hat{e}_b} \Phi = (\hat{e}_b)^\mu \partial_\mu \Phi = \partial_b \Phi & \text{for } B = b \leq d \\ 0 & \text{for } B = b + d \end{cases}$$

since the vector part of E^b is null, and where $\partial_b = (\hat{e}_b)^\mu \partial_\mu$. We also note that the embedding of $\mathrm{d}\phi$ in E is equal to

$$\frac{1}{2}(\partial^A \phi)\phi^{-1} \hat{E}_A = \frac{1}{2}(\partial^{A=d+a}\phi)(\phi^{-1} E^a) = \frac{1}{2} 2 (\partial_a \phi) e^a = (\partial_a \phi) e^a = \mathrm{d}\phi,$$

recalling that $\eta^{AB} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Coming back to the Lie derivative of Φ we have:

$$L_{\Phi^{-1}\hat{E}_b} \Phi = \partial_b \Phi = \partial_b(e^{-2\phi}(\det e)) = e^{-2\phi}(\det e)\ \mathrm{tr}(e^{-1}\partial_b e) - 2(\partial_b \phi)\Phi,$$

where $e^b{}_\nu = (e^b)_\nu$ and $(e^{-1})^\nu{}_b = (\hat{e}_b)^\nu$. The first term is equal to

$$\Phi \hat{e}_a{}^\nu \partial_b (e^a{}_\nu) = \Phi \hat{e}_a{}^\nu \hat{e}_b{}^\mu \partial_\mu (e^a{}_\nu) = \Phi i_{\hat{e}_a} i_{\hat{e}_b} \mathrm{d}e^a = -\Phi i_{\hat{e}_b} i_{\hat{e}_a} \mathrm{d}e^a,$$

where in coordinate components you have $\mathrm{d}((e^b)_\nu \mathrm{d}x^\nu) = \partial_\mu((e^b)_\nu) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

The second term is equal to $\partial_b(e^{-2\phi})(\det e) = -2(\partial_b \phi)\Phi = -2\Phi i_{\hat{e}_b} \mathrm{d}\phi$, hence the result

$$L_{\Phi^{-1}\hat{E}_b} \Phi = -\Phi(i_{\hat{e}_b} i_{\hat{e}_d} \mathrm{d}e^d + 2 i_{\hat{e}_b} \mathrm{d}\phi).$$

□

On the other hand,

$$L_{\Phi^{-1}\hat{E}_B}\Phi^{-1}\hat{E}_C = \begin{pmatrix} [\hat{e}_b, \hat{e}_c] + i_{([\hat{e}_b,\hat{e}_c]}B - i_{\hat{e}_b}i_{\hat{e}_c}H & \mathcal{L}_{\hat{e}_b}e^c \\ -\mathcal{L}_{\hat{e}_c}e^b & 0 \end{pmatrix}_{BC},$$

where $H = dB$.

Proof. We will be using Cartan's formula: $\mathcal{L}_v\lambda = \mathrm{d}(i_v\lambda) + i_v(\mathrm{d}\lambda)$ for λ a differential form.

Writing the E split conformal basis elements $\Phi^{-1}\hat{E}_B = V = v + \lambda$ and $\Phi^{-1}\hat{E}_C = W = w + \zeta$, we calculate $L_{\Phi^{-1}\hat{E}_B}\Phi^{-1}\hat{E}_C = L_V W = \mathcal{L}_v w + \mathcal{L}_v\zeta - i_w\mathrm{d}\lambda$ for the different cases:

- $B = d + b$, $C = c + d$. $V = e^b$, $W = e^c$, so $v = 0$ and $w = 0$, therefore $L_V W = 0$;
- $B = b \leq d$, $C = c + d$. $V = \hat{e}_b + i_{\hat{e}_b}B$, $W = e^c$, so $w = 0$ and $L_V W = \mathcal{L}_v\zeta = \mathcal{L}_{\hat{e}_b}e^c$;
- $B = b + d$, $C = c \leq d$. $V = e^b$, $W = \hat{e}_c + i_{\hat{e}_c}B$, so $v = 0$ and

$$L_V W = -i_w d\lambda = -i_{\hat{e}_c}\mathrm{d}e^b = -\mathcal{L}_{\hat{e}_c}e^b + d(i_{\hat{e}_c}e^b) = -\mathcal{L}_{\hat{e}_c}e^b + \mathrm{d}(\delta^b_c) = -\mathcal{L}_{\hat{e}_c}e^b;$$

- $B = b \leq d$, $C = c \leq d$. $V = \hat{e}_b + i_{\hat{e}_b}B$, $W = \hat{e}_c + i_{\hat{e}_c}B$, so

$$\begin{aligned} L_V W &= \mathcal{L}_{\hat{e}_b}\hat{e}_c + \mathcal{L}_{\hat{e}_b}(i_{\hat{e}_c}B) - i_{\hat{e}_c}\mathrm{d}(i_{\hat{e}_b}B) \\ &= [\hat{e}_b, \hat{e}_c] + \mathcal{L}_{\hat{e}_b}(i_{\hat{e}_c}B) - i_{\hat{e}_c}\mathrm{d}(i_{\hat{e}_b}B) \end{aligned}$$

  We note the property $i_{[X,Y]} = [\mathcal{L}_X, i_Y]$, so

$$i_{[\hat{e}_b,\hat{e}_c]}B = \mathcal{L}_{\hat{e}_b}(i_{\hat{e}_c}B) - i_{\hat{e}_c}(\mathcal{L}_{\hat{e}_b}B) = \mathcal{L}_{\hat{e}_b}(i_{\hat{e}_c}B) - i_{\hat{e}_c}(\mathrm{d}(i_{\hat{e}_b}B) + i_{\hat{e}_b}(\mathrm{d}B)),$$

  and we conclude:

$$L_V W = [\hat{e}_b, \hat{e}_c] + i_{[\hat{e}_b,\hat{e}_c]}B + i_{\hat{e}_c}i_{\hat{e}_b}(\mathrm{d}B) = [\hat{e}_b, \hat{e}_c] + i_{[\hat{e}_b,\hat{e}_c]}B - i_{\hat{e}_b}i_{\hat{e}_c}H,$$

  since $i_{\hat{e}_c}i_{\hat{e}_b}$ is antisymmetric in b and c.

□

Summing the torsion and the third term, which we calculated in 1) and 2) respectively, with $W = w + \lambda$ and $V = v + \zeta$, we obtain

$$\begin{gathered} \left[(T_1)^A{}_{BC} + (T_2)_B\delta^A_C\right] V^B W^C \hat{E}_A = \\ -v^b W^C \delta^A_C (i_{\hat{e}_b}i_{\hat{e}_d}\mathrm{d}e^d + 2i_{\hat{e}_b}\mathrm{d}\phi)\hat{E}_A + v^b w^c \left([\hat{e}_b, \hat{e}_c]^a \hat{E}_a - (i_{\hat{e}_b}i_{\hat{e}_c}H)_a E^a\right) \\ +(\mathcal{L}_{\hat{e}_b}e^c)_a E^a v^b \zeta_c + \lambda_b w^c (\omega_c{}^b{}_a - \omega_a{}^b{}_c - \mathcal{L}_{\hat{e}_c}e^b)_a E^a. \end{gathered}$$

We recognise that $(T_2)_B V^B$ must be $-v^b(i_{\hat{e}_b}i_{\hat{e}_d}\mathrm{d}e^d + 2i_{\hat{e}_b}\mathrm{d}\phi)$, so

$$(T_2)_B = \begin{cases} -(i_{\hat{e}_b}i_{\hat{e}_d}\mathrm{d}e^d + 2i_{\hat{e}_b}\mathrm{d}\phi) & \text{for } B = b \leq d, \\ 0 & \text{for } B > d. \end{cases}$$

We can deduce that T_2^B with index raised is equal to

$$(T_2)^B = \eta^{BA}(T_2)_B = \begin{cases} 0 & \text{for } B = b \leq d \\ -2(i_{\hat{e}_b}i_{\hat{e}_d}\mathrm{d}e^d + 2i_{\hat{e}_b}\mathrm{d}\phi) & \text{for } B > d. \end{cases}$$

Taking $\{\hat{e}_a\}$ to be the coordinate basis, only the $\mathrm{d}\phi$ term remains since $\mathrm{d}\mathrm{d}x^\mu = 0$, and we have that T_2 is the embedding of $-4(i_{\partial_\mu}\mathrm{d}\phi)\mathrm{d}x^\mu = -4\mathrm{d}\phi$ in E.

We deduce T_1 from the remaining terms of the sum:

$$(T_1)^A{}_{BC} = \begin{cases} \text{for } A = a \leq d, \begin{cases} \text{for } B = b \leq d, C = c \leq d : [\hat{e}_b, \hat{e}_c]^a, \\ \text{otherwise: } 0, \end{cases} \\ \text{for } A = a + d, \begin{cases} B = b,\ C = c : -(i_{\hat{e}_b} i_{\hat{e}_c} H)_a, \\ B = b,\ C = c + d :\ (\mathcal{L}_{\hat{e}_b} e^c)_a, \\ B = b + d,\ C = c :\ -(\mathcal{L}_{\hat{e}_c} e^b)_a + \omega_c{}^b{}_a - \omega_a{}^b{}_c, \\ B = b + d,\ C = c + d :\ 0. \end{cases} \end{cases}$$

If we take the coordinate basis, only the H term remains. Indeed, in coordinate indices the components of the torsion are $\omega_\nu{}^\mu{}_\lambda - \omega_\lambda{}^\mu{}_\nu = T^\mu{}_{\nu\lambda}$, therefore this term cancels as we are using a torsion-free connection. Regarding the other terms, we have $[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}] = 0$ and $\mathcal{L}_{\frac{\partial}{\partial x^\mu}} \mathrm{d}x^\nu = 0$: partial derivatives commute and $\mathcal{L}_{\frac{\partial}{\partial x^\mu}} \mathrm{d}x^\nu = \mathrm{d}(i_\mu \mathrm{d}x^\nu) + i_\mu(\mathrm{d}^2 x^\mu) = \mathrm{d}(\delta^\nu_\mu) + 0 = 0$.

Thus we have

$$(T_1)^M{}_{NP} = \begin{cases} -(i_{\partial/\partial x^\nu} i_{\partial/\partial x^\lambda} H)_\mu & \text{for } M = \mu + d,\ N = \nu \leq d,\ P = \lambda \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Raising the indices B and C, we obtain

$$(T_1)^{MNP} = \begin{cases} -2^2(i_{\partial/\partial x^\nu} i_{\partial/\partial x^\lambda} H)_\mu = -4 i_{\partial/\partial x^\mu} i_{\partial/\partial x^\nu} i_{\partial/\partial x^\lambda} H = -4H_{\mu\nu\lambda} & \text{for } M = \mu + d,\ N = \nu + d,\ P = \lambda + d, \\ 0 & \text{otherwise,} \end{cases}$$

which is the embedding of $-4H$ in E.

Finally we have the coordinate frame components of the generalised torsion of D^∇ with a torsion-free conventional connection ∇:

$$T_1 = -4H, \quad T_2 = -4\mathrm{d}\phi,$$

where we are using the embedding $T^*M \to E$ to be able to write the E objects in terms of differential forms.

3.4.4 The absence of generalised curvature: conditional tensoriality

We would like to introduce a form of generalised curvature on E for a given generalised connection D in analogy to the usual definition $R(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w$, by replacing ∇ with D and the Lie bracket with the Courant bracket:

$$R(U, V, W) = [D_U, D_V]W - D_{[\![U,V]\!]}W.$$

Remark. $(U, V) \to R(U, V, .)$*, lives in* $\Gamma((E \otimes E) \otimes o(d, d))$*, since the final object acts on* W *via the adjoint representation of* $O(d, d)$*.*

However, this object is non-tensorial.

If R were tensorial we would have $R(U, V, W)^M = R^M_{NPL} U^N V^P W^L$, making R linear in U, V and W. Checking for linearity in the arguments U and V, we have for some scalar functions f, g

$$R(fU, gV, W) = [D_{fU}, D_{gV}]W - D_{[\![fU,gV]\!]}W = fg\left([D_U, D_V]W - D_{[\![U,V]\!]}W\right) - \frac{1}{2}\langle U, V\rangle D_{(fdg - gdf)}W.$$

The curvature is not linear in U and V: In the last term, f, g don't present themselves as scalar factors.

Proof.

$$[D_{fU}, D_{gV}]W = [fD_U, gD_V]W = fg[D_U, D_V]W + fD_U(g)D_V W - gD_V(f)D_U W$$

where $D_U f = U^M \partial_M f$.

$$D_{[\![fU,gV]\!]} = \left(fU^N\partial_N(gV^M) - gV^N\partial_N(fU^M) - \frac{1}{2}\left(fU_N\partial^M(gV^N) - gV_N\partial^M(fU^N)\right)\right) D_M W$$

$$= fgD_{[\![U,V]\!]}W + \left(\partial_N(g)fU^N V^M - g\partial_N(f)V^N U^M - \frac{1}{2}(f\partial^M g - g\partial^M f)U_N V^N\right) D_M W$$

$$= fgD_{[\![U,V]\!]}W + (D_U g)fD_V W - g(D_V f)D_U W - \langle U, V\rangle D_{f\mathrm{d}g - g\mathrm{d}f}W$$

where the exterior derivative of a function $\mathrm{d}f$ embedded in E is equal to $\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu = \frac{1}{2}(\partial^M f)\hat{E}_M$, where $\{\hat{E}_M\}$ is the coordinate basis in E. *Error: supposed to be plus on last term and factor of a half.* □

With additional structure, we can define more constrained objects that are tensorial measures of general curvature: Let $C_1 \in E$ and $C_2 \in E$ be subspaces such that $\langle U, V\rangle = 0$ for all $U \in \Gamma(C_1)$ and $V \in \Gamma(C_2)$. Then the final term in the expression of $R(fU, gV, W)$ vanishes and R is linear in all arguments. It can be proven that $R \in \Gamma((C_1 \otimes C_2) \otimes o(d, d))$ is a tensor.

4 Supergravity in Generalised Geometry

We would now like to construct the generalised analogue of the Levi-Civita connection: the name for the unique torsion-free conventional connection that preserves the $O(d) \subset GL(d, (R))$ structure defined by a metric g.

Here, in generalised geometry and in the context of supergravity, we are interested in generalised connections preserving an $O(p, q) \times O(p, q) \subset O(d, d) \times \mathbb{R}^+$ structure on $\tilde{F}$, where $p + q = d$.

We will find that it is possible to construct torsion-free connections of this type, but there is no unique choice.

4.1 $O(p, q) \times O(p, q)$ structures and the generalised metric

Consider an $O(p, q) \times O(q, p) \subset O(d, d) \times \mathbb{R}^+$ principal sub-bundle P of the generalised structure bundle $\tilde{F}$.

We will see that specifying such a sub-bundle is equivalent to specifying a conventional metric g of signature (p, q), a B-field patched as in supergravity (or as in a split frame), and a dilaton ϕ: all the elements needed to capture the NSNS supergravity fields.

Geometrically, an $O(p, q) \times O(q, p)$ structure

- fixes a nowhere vanishing section $\Phi \in \Gamma(\det T^*M)$ since it is in fact a subgroup of $O(d, d)$, giving an isomorphism between weighted and unweighted generalised tangent spaces $\tilde{E}$ and E.
- defines a splitting of E into two d-dimensional sub-bundles: by property of G-modules, each fibre vector space E_x will be split into a direct sum $E_x = (C_+)_x \oplus (C_-)_x$, with the first being an $O(p, q)$-module, and the second an $O(q, p)$ module.

  We can write
  $$E = C_+ \oplus C_-,$$
  such that the $O(d, d)$ metric η restricts to two separate metrics, one of signature (p, q) on C_+ and one of signature (q, p) on C_-. Each sub-bundle is isomorphic to TM.

We can identify a special set of frames defining an $O(p,q) \times O(q,p)$ bundle, isomorphic to a sub-bundle of $\tilde{F}$ as we defined it. An $O(p,q)\times O(q,p)$ sub-bundle is a subset of frames that can be entirely accessed from one frame via $O(p,q) \times O(q,p)$ group element transformations or a representation of these; we describe her as special a choice of one of these subsets where we have a conserved η form which manifests the $O(p,q) \times O(q,p)$ symmetry.

To have this manifest $O(p,q) \times O(q,p)$ symmetry, we define a frame $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$ such that $\{\hat{E}^+_a\}$ form an orthonormal frame for C_+ and $\{\hat{E}^-_{\bar{a}}\}$ for C_-; the union of the two forms a frame of E. By definition these frame elements satisfy:

$$\begin{cases} \left\langle \hat{E}^+_a, \hat{E}^+_b \right\rangle = \Phi^2 \eta_{ab}, \\ \left\langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \right\rangle = -\Phi^2 \eta_{\bar{a}\bar{b}}, \\ \left\langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \right\rangle = 0, \end{cases}$$

where the inner product symbol correspond to the η metric contraction as usual, $\Phi \in \Gamma(\det T^*M)$ is now some fixed density, and η_{ab} and $\eta_{\bar{a}\bar{b}}$ represent the same flat metric with signature (p,q), with possibly different forms due to different bases (barred and unbarred).

Remark. *$\eta_{ab} \mapsto -\eta_{ab}$ gives an isomorphism between metrics of signature (p,q) and $-(p,q) = (q,p)$.*

$O(p,q)$ denotes the group that preserves the form of a given metric of signature (p,q). We note that we can be this general in our definition since different metrics of same signature are isomorphic to one another - in the sense that you go from one form to the other with a change of basis, $A^{-1}\eta A = \eta'$ - so the two groups defined by preserving one form and the other form respectively are isomorphic, and you can use A to get from one set of transformation matrices to the other.

The corresponding conformal frame

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & for A = a \\ \hat{E}^-_{\bar{a}} & for A = \bar{a} + d \end{cases}$$

satisfies

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \Phi^2 \eta_{AB}, \text{ where } \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}.$$

We note that η_{AB}, while being the same metric, has a different form than before; this is the form we will be referring to throughout this section. We redefine orthonormal to be the case $\Phi = 1$ of the above of conformal frames.

Remark. *It is clear that we have $\forall V \in \Gamma(C_-),\ W \in \Gamma(C_+),\ \langle V, W \rangle = 0$. We will attempt to use this result in the Lichnerowicz section.*

We will use the convention of raising and lowering small indices with their corresponding metrics η_{ab} and $\eta_{\bar{a}\bar{b}}$, and $2d$ - dimensional capital letter indices with the $O(d,d)$ metric η_{AB}. For instance, we have

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{for } A = a \\ -\hat{E}^{-a} & \text{for } A = \bar{a} + d. \end{cases}$$

One can write a generic $O(p,q) \times O(q,p)$ structure explicitly as

$$\begin{cases} \hat{E}^+_a = e^{-2\phi}\sqrt{-g}(\hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B), \\ \hat{E}^-_{\bar{a}} = e^{-2\phi}\sqrt{-g}(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B), \end{cases}$$

where $e^+_a = \eta_{ab} e^{+b}$, $e^-_{\bar{a}} = \eta_{\bar{a}\bar{b}} e^{-\bar{b}}$ and $\Phi = e^{-2\phi}\sqrt{-g}$ is the fixed conformal factor; $\{\hat{e}^+_a\}$, $\{\hat{e}^-_{\bar{a}}\}$ are two independent orthonormal frames for the metric g (same as their duals for the inverse of g):

$$g = \eta_{ab} e^{+a} \otimes e^{+b} = \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}},$$

or equivalently

$$g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, \quad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}.$$

Remark.

- *$\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$ are both independent frames of TM; these bases have nothing to do with the isomorphism of both C_+ and C_- with TM, and one must be careful not to confuse $\hat{e}^{+/-}_a$ with $\hat{E}^{+/-}_a$.*
- *We have $e^+_a = \eta_{ab} e^{+b} \in E^*$ which we can see from the index being down but there being no hat on the e, as we remember η brings E to its dual space. However we regard $e^{+/-}_a$ as an element of E, and more precisely as a cotangent vector embedded in E by being a linear combination of the $\{e^b\}$ with coefficients η_{ab}.*

Proof. Here we will prove that this form of conformal frame corresponds to a generic $O(p,q)\times O(q,p)$ structure. We will use the following properties:

- As metrics, η_{ab} and $\eta_{\bar{a}\bar{b}}$ are symmetric;
- The interior product satisfies $\{i_X, i_Y\} = 0$ by antisymmetry of forms;
- For $\{\hat{e}_a\}$ a basis on the tangent fibre and $\{e^a\}$ its dual, $(e^a)_b(\hat{e}_a)^c = \delta^c_b$ by definition of the dual basis;
- The inner product is bilinear, and we already know $\langle \hat{e}_a + i_{\hat{e}_a}B, \hat{e}_b + i_{\hat{e}_b}B\rangle = 0$ - this provides a shortcut for the calculations in the first and second of the following cases.

We have

$$\begin{cases} \left\langle \hat{E}^+_a, \hat{E}^+_b \right\rangle = \Phi^2 \frac{1}{2}(i_{\hat{e}^+_a} e^+_b + i_{\hat{e}^+_b} e^+_a) = \Phi^2 \frac{1}{2}\left(i_{\hat{e}^+_a}(\eta_{bc} e^{+c}) + i_{\hat{e}^+_b}(\eta_{ac} e^{+c})\right) = \Phi^2 \frac{1}{2}(\eta_{ba} + \eta_{ab}) = \Phi^2 \eta_{ab}, \\ \left\langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \right\rangle = -\Phi^2 \frac{1}{2}(i_{\hat{e}^-_{\bar{a}}} e^-_{\bar{b}} + i_{\hat{e}^-_{\bar{b}}} e^-_{\bar{a}}) = -\Phi^2 \eta_{\bar{a}\bar{b}} \text{ in the same way,} \\ \left\langle \hat{E}^-_{\bar{a}}, \hat{E}^+_b \right\rangle = i_{\hat{e}^-_{\bar{a}}}(e^+_b + i_{\hat{e}^+_b} B) + i_{\hat{e}^+_b}(-e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B) \\ \qquad = \eta_{bc}(i_{\hat{e}^-_{\bar{a}}} e^{+c}) - \eta_{\bar{a}\bar{c}} i_{\hat{e}^+_b} e^{-\bar{c}} + i_{\hat{e}^-_{\bar{a}}} i_{\hat{e}^+_b} B + i_{\hat{e}^+_b} i_{\hat{e}^-_{\bar{a}}} B = \eta_{bc}(\hat{e}^-_{\bar{a}})^d (i_{\hat{e}^+_d} e^{+c}) - \eta_{\bar{a}\bar{c}} i_{\hat{e}^+_b} e^{-\bar{c}} \\ \qquad = \eta_{bc}(\hat{e}^-_{\bar{a}})^d \delta^c_d - \left((\hat{e}^-_{\bar{a}})^d \eta_{de} (\hat{e}^-_{\bar{c}})^e\right)(e^{-\bar{c}})_f i_{\hat{e}^+_b} e^{+f} = \eta_{bc}(\hat{e}^-_{\bar{a}})^c - (\hat{e}^-_{\bar{a}})^d \eta_{de} \delta^e_f \delta^f_b \\ \qquad = \eta_{bc}(\hat{e}^-_{\bar{a}})^c - \eta_{bc}(\hat{e}^-_{\bar{a}})^c = 0 \end{cases}$$

□

By this explicit construction of an $O(p,q) \times O(q,p)$ structure using split frames, where as we know B absorbs the particular patching on E while all other elements it is summed with can be globally defined, we see that there is no E-patching, topological obstruction to the existence of such structures.

One can see that specifying an $O(p,q) \times O(q,p)$ sub-bundle is equivalent to specifying a conventional metric g of signature (p,q), a B-field patched as in supergravity, and a dilaton ϕ: As we can see from their appearances in the generic structure, fixing these leaves only (and rightly so) the freedom to go from one frame to another via a matrix $M \in O(p,q) \times O(q,p)$, where the first factor acts on $\{\hat{e}^+_a\}$ (and inversely its dual), the second on $\{\hat{e}^-_{\bar{a}}\}$.

We can combine the invariance of B and g under this subgroup in an invariant generalised metric G, with the form

$$G = \Phi^{-2}\left(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}\right), \quad i.e.\ \ G_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix},$$

which in the coordinate frame takes the expression

$$G_{MN} = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}_{MN}.$$

G encodes both g and B: The bottom right corner gives you g; you can then deduce B from any other corner. Thus, by construction, G and Φ together specify a subset of $\tilde{F}$ with an $O(p,q)\times O(q,p)$ structure, which is equivalent to saying that the pair (G,Φ) parametrise the coset $(O(d,d)\times \mathbb{R}^+)/(O(p,q)\times O(q,p))$ where $p+q=d$.

4.2 Torsion-free, compatible connections

A generalised connection D is compatible with the $O(p,q)\times O(q,p)$ structure $P\subset\tilde{F}$ if

$$DG = 0, \quad D\Phi = 0,$$

meaning that D acts within the $(O(d,d)\times\mathbb{R}^+)/(O(p,q)\times O(q,p))$ coset or sub-bundle, so that for $W\in\Gamma(\tilde{E})$ given by

$$W = w^a_+\hat{E}^+_a + w^{\bar{a}}_-\hat{E}_{\bar{a}^-},$$

we have

$$D_M W^A = \begin{cases} \partial_M w^a_+ + \Omega_M{}^a{}_b w^b_+ & \text{for } A = a \\ \partial_M w^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_b w^{\bar{b}}_- & \text{for } A = \bar{a} \end{cases}$$

with

$$\Omega_{Mab} = -\Omega_{Mba}, \quad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}, \text{ i.e. both connections have an } O(d,d) \text{ adjoint action.}$$

Indeed, we saw that an $O(p,q)\times O(q,p)$ structure is one that fixes a nowhere vanishing section Φ, and defines a splitting of E into $E = C_+\oplus C_-$ such that the $O(d,d)$ metric restricts to a separate metric of signature (p,q) on C_+ and a metric of signature (q,p) on C_-. This is exactly what is encoded in this connection: the fixing of the dilaton Φ reduces the adjoint action $\tilde{\Omega}$ to an $O(d,d)$ adjoint action Ω, and finally the separate $O(d,d)$ adjoint actions on each of the two sub-bundles C_+ and C_- is equivalent to an $O(p,q)\times O(q,p)$ adjoint action on E.

In this subsection we will prove the following theorem:

Theorem. *Given an $O(p,q)\times O(q,p)$ structure $P\subset\tilde{F}$, there always exists a torsion-free, compatible generalised connection D. However, it is not unique.*

We will start by constructing a compatible connection.

Let ∇ be the Levi-Civita connection for the metric g: ∇ is torsion free, and $\nabla g = 0$ or equivalently, for a given basis $\{\hat{e}_a\}$, $\nabla_{\frac{\partial}{\partial x^\mu}}\hat{e}_a = \omega_\mu{}^b{}_a$ where the set of connection one-forms $\omega^a{}_b$ take values in the adjoint representation of $O(p,q)$, the group preserving g.

In terms of the two orthonormal bases $\{\hat{e}_a\}$ and $\{\hat{e}_{\bar{a}}\}$, we get two gauge equivalent spin-connections: if $v = v^a\hat{e}^+_a = v^{\bar{a}}\hat{e}^-_{\bar{a}}\in\Gamma(TM)$ we have

$$\nabla_\mu v^\nu = (\partial_\mu v^a + \omega^{+a}_\mu{}_b v^b)(\hat{e}^+_a)^\nu = (\partial_\mu v^{\bar{a}} + \omega_\mu{}^{-\bar{a}}{}_{\bar{b}} v^{\bar{b}})(\hat{e}_{\bar{a}^-})^\nu$$

$$= (\nabla_\mu v_+^a)(\hat{e}_a^+)^\nu = (\nabla_\mu v_-^{\bar{a}})(\hat{e}_{\bar{a}}^-)^\nu.$$

We can then define, similarly to before but with coefficients in a different frame,

$$D_M^{\nabla(2)} W^a = \begin{cases} \nabla_\mu w_+^a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}, \quad D_M^{\nabla(2)} W^{\bar{a}} = \begin{cases} \nabla_\mu w_-^{\bar{a}} & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d, \end{cases}$$

By property of the Levi-Civita connection, $\omega^+_{\mu ab} = -\omega^+_{\mu ba}$ and $\omega^-_{\mu ab} = -\omega^-_{\mu ba}$, therefore this generalised connection is compatible with an $O(p,q) \times O(q,p)$ structure.

However, we can calculate the torsion of $D^{\nabla(2)}$ and show that it is not torsion-free. If we choose the two orthonormal frames to be aligned, so $\hat{e}_a^+ = \hat{e}_{\bar{a}}^- = \hat{e}_a$ and therefore $e_a^+ = e_a^- = e_a$, we have

$$\begin{aligned} W &= w_+^a \hat{E}_a^+ + w_-^{\bar{a}} \hat{E}_{\bar{a}}^- = w_+^a \Phi(\hat{e}_a + e_a + i_{\hat{e}_a} B) + w_-^a \Phi(\hat{e}_a - e_a + i_{\hat{e}_a} B) \\ &= (w_+^a + w_-^a)\hat{E}_a + (w_+^a - w_-^a)\eta_{ab} e^b = (w_+^a + w_-^a)\hat{E}_a + (w_{+a} - w_{-a})E^a, \end{aligned}$$

where $\{\hat{E}_a\} \cup \{E^a\}$ is the conformal split frame with the same B, Φ and $\{\hat{e}_a\}$. This alignment also implies that the action of a connection on a plus coordinate is equal to that on a minus coordinate: $\omega_{\mu\ b}^{+a} = \omega_{\mu\ \bar{b}}^{-\bar{a}}$ so $\nabla_\mu w_+^a = \nabla_\mu w_-^a$, and since we have

$$\begin{cases} (\nabla_\mu w_+^a + \nabla_\mu w_-^a) = \nabla_\mu (w_+^a + w_-^a) \\ \eta_{ab}(\nabla_\mu w_+^b - \nabla_\mu w_-^b) = \nabla_\mu (w_{+a} - w_{-a}) \text{ as } \nabla \eta_{ab} = 0. \end{cases}$$

We can conclude that when we choose the two orthonormal frames to coincide, our definition of the connection $D_M^{\nabla(2)}$ agrees with our definition of $D_M^{\nabla(1)}$. Seeing as the Levi-Civita connection is torsion-free by definition, we can therefore use our previous calculations of the generalised torsion for $D_M^{\nabla(1)}$ with ∇ torsion-free, to calculate the torsion here of $D_M^{\nabla(2)}$. We have the non-zero generalised torsion components

$$T_1 = -4H, \quad T_2 = -4\mathrm{d}\phi.$$

Remark. *As the torsion is a tensor, this is valid for any $O(p,q)$ bases $\{\hat{e}_a\}$ and $\{\hat{e}_{\bar{a}}\}$, aligned or not.*

By definition and form of a connection, we note that a generalised connection D can always be written as

$$D_M W^A = D_M^{\nabla(2)} W^A + \Sigma_M{}^A{}_B W^B.$$

If D is compatible with the $O(p,q) \times O(q,p)$ structure, then we have:

- $\Sigma_M{}^a{}_{\bar{b}} = \Sigma_M{}^{\bar{a}}{}_b = 0$, since D acting only in the sub-bundle is equivalent to the cross-terms of $\Sigma + \Omega$ being null; but Ω already individually satisfies this property, which imposes that Σ must as well.

- $\Sigma_{Mab} = -\Sigma_{Mba}$, $\Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}}$, which is the $O(d,d)$ adjoint action condition on each of the sub-bundle connections, already satisfied by Ω.

By definition and tensor linearity, the generalised torsion components of D are given by

$$(T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]}, \quad (T_2)_A = -4\mathrm{d}\phi_A - \Sigma_C{}^C{}_A,$$

where $\mathrm{d}\phi^A$, and H^{ABC} are the components in frame indices of the corresponding forms $\mathrm{d}\phi$ and H under the embeddings $T^*M \to E$ and $\Lambda^3 T^*M \to \Lambda^3 E^*$ respectively. Their indices are lowered by η_{AB} as usual, bringing these objects into E^*. We want to write the covariant derivative in terms of small-index $a, \bar{a}$ components of its different elements. To do so, we first write $\mathrm{d}x^\mu$ embedded in E^*, in the dual frame:

$$\mathrm{d}x^\mu = \frac{1}{2}\Phi^{-1}(\hat{e}_a^{+\mu} \hat{E}^{+a} - \hat{e}_{\bar{a}}^{-\mu} \hat{E}^{-\bar{a}}).$$

Proof. The right-hand side is equal to

$$\frac{1}{2}\hat{e}_a^{+\mu}(\eta^{ab}\hat{e}_b^{+} + e^{+a} + \eta^{ab} i_{\hat{e}_b^{+}} B) + \frac{1}{2}\hat{e}_{\bar{a}}^{-\mu}(-\eta^{\bar{a}\bar{b}}\hat{e}_{\bar{b}}^{-} + e^{-\bar{a}} - \eta^{\bar{a}\bar{b}} i_{\hat{e}_{\bar{b}}^{-}} B).$$

Writing $\hat{e}_a$ for either of the orthonormal bases, we have

$$\hat{e}_a^{\mu}\eta^{ab}(\hat{e}_b^{\nu}\frac{\partial}{\partial x^{\nu}}) = \eta^{\mu\nu}\frac{\partial}{\partial x^{\nu}},$$

$$\hat{e}_a^{\mu}\eta^{ab}\hat{e}_b^{\nu} i_{\frac{\partial}{\partial x^{\mu}}} B = \eta^{\mu\nu} i_{\frac{\partial}{\partial x^{\mu}}} B,$$

$$\hat{e}_a^{\mu}\eta^{ab}(e_{\nu}^{a}\mathrm{d}x^{\nu}) = \delta_{\nu}^{\mu}\mathrm{d}x^{\nu} = \mathrm{d}x^{\mu}.$$

Replacing the terms in our expression we find that the right-hand side is indeed equal to $\frac{1}{2}\times 2\mathrm{d}x^{\mu} = \mathrm{d}x^{\mu}$. □

Now we can easily express $\mathrm{d}\phi$ in the dual frame $\Phi^{-1}\hat{E}^{A} = \begin{cases} \Phi^{-1}\hat{E}^{+a} & \text{for } A = a \\ -\Phi^{-1}\hat{E}^{-\bar{a}} & \text{for } A = \bar{a} + d \end{cases}$.

Using $(\partial_{\mu}\phi)\hat{e}_a^{\mu} = \partial_a\phi$ for a generic tangent frame $\{\hat{e}_a\}$, we have

$$\mathrm{d}\phi = \partial_{\mu}\phi\mathrm{d}x^{\mu} = \mathrm{d}\phi = \frac{1}{2}\partial_a\phi(\Phi^{-1}\hat{E}^{+a}) - \frac{1}{2}\partial_{\bar{a}}\phi(\Phi^{-1}\hat{E}^{-\bar{a}}),$$

giving us the dual frame components

$$\mathrm{d}\phi_A = \begin{cases} \frac{1}{2}\partial_a\phi & A = a \\ \frac{1}{2}\partial_{\bar{a}}\phi & A = \bar{a} + d. \end{cases}$$

We can similarly write H in frame indices, with the embedding and decomposition

$$\Lambda^3 T^*M \to \Lambda^3 E = \Lambda^3(C_+ \oplus C_-) \simeq \Lambda^3 C_+ \oplus (\Lambda^2 C_+ \otimes C_-) \oplus (C_+ \otimes \Lambda^2 C_-) \oplus \Lambda^3 C_-.$$

Remark. *There is no antisymmetry in an E wedge product of an element of C_+ with an element of C_-, since the two indices run over two distinct subsets of E indices, and therefore can never be exchanged. Thus, only the above terms remain in the decomposition of $\Lambda^3 E = \Lambda^3(C_+ \oplus C_-)$ into a direct sum.*

We want to solve

$$H = (H)_{\mu\nu\lambda}\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\lambda} = H_{ABC}\hat{E}^{A} \wedge \hat{E}^{B} \wedge \hat{E}^{C}.$$

Inserting the expression for $\mathrm{d}x^{\mu}$, we find

$$H_{\mu\nu\lambda}\hat{e}_a^{+\mu}\hat{e}_b^{+\nu}\hat{e}_c^{+\lambda}(\frac{1}{2})^3(\Phi^{-1}\hat{E}^{+a}) \wedge (\Phi^{-1}\hat{E}^{+b}) \wedge (\Phi^{-1}\hat{E}^{+c})$$

$$+H_{\mu\nu\lambda}e_a^{+\mu}e_b^{+\nu}e_{\bar{c}}^{-\lambda}(\frac{1}{2})^3(\Phi^{-1}\hat{E}^{+a}) \wedge (\Phi^{-1}\hat{E}^{+b}) \wedge (-\Phi^{-1}\hat{E}^{-\bar{c}}) + ...,$$

where the other terms follow the same pattern with respect to barred versus unbarred indices. Note that for a given cotangent frame $\{e^a\}$, we have $(\hat{e}_a)^{\mu}(e^a)_{\nu} = \delta_{\nu}^{\mu}$ by definition of the dual basis, which implies $(\hat{e}_a)^{\mu}e^a = \mathrm{d}x^{\mu}$. So for some one-form v we have $v = v_{\mu}\mathrm{d}x^{\mu} = v_{\mu}(\hat{e}_a)^{\mu}e^a$, and its frame components are $v_a = v_{\mu}(\hat{e}_a)^{\mu}$. This extends of course to any differential form, and we have $H_{\mu\nu\lambda}\hat{e}_a^{\mu}\hat{e}_b^{\nu}\hat{e}_c^{\lambda} = H_{abc}$, where each index can belong to a different frame, be barred or unbarred, and its frame element is marked with the corresponding sign $+/-$.

Finally, we have the components

$$H_{ABC} = \begin{cases} \frac{1}{8} H_{abc} & (A,B,C) = (a,b,c) \\ \frac{1}{8} H_{ab\bar{c}} & (A,B,C) = (a,b,\bar{c}+d) \\ \frac{1}{8} H_{a\bar{b}\bar{c}} & (A,B,C) = (a,\bar{b}+d,\bar{c}+d) \\ \frac{1}{8} H_{\bar{a}\bar{b}\bar{c}} & (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d). \end{cases}$$

Due to the form of Σ in the connection, we set its natural components to be defined with the middle index up, the other two down:

$$\Sigma_{A\ B}^{\ C} \hat{E}^A = \begin{cases} \Sigma_{a\ c}^{\ b} & \text{for } A = a \leq d, B = b \leq d, C = c \leq d, \\ \Sigma_{\bar{a}\ c}^{\ b} & \text{for } A = a+d, B = b \leq d, C = c \leq d, \\ etc. \end{cases}$$

with $\Sigma_{A\ \bar{c}}^{\ b} = \Sigma_{A\ c}^{\ \bar{b}} = 0$. So we have

$$\Sigma_{ABC} = \begin{cases} \Sigma_{AbC} & \text{for } B = b,\ A = a \text{ or } \bar{a},\ C = c \text{ or } \bar{c}, \\ -\Sigma_{A\bar{b}C} & \text{for } B = b,\ A = a \text{ or } \bar{a},\ C = c \text{ or } \bar{c}. \end{cases}$$

Now that we have the dual conformal frame components of $\mathrm{d}\phi$, B and Σ, we want to deduce what setting the torsion of D to zero translates to in terms of these components.

$$\begin{cases} (T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]} = 0 \\ (T_2)_A = -4\mathrm{d}\phi_A - \Sigma_{C\ A}^{\ C} = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \Sigma_{[abc]} = -\frac{4}{3}(\frac{1}{8}H_{abc}) = -\frac{1}{6}H_{abc}, \\ \Sigma_{[\bar{a}\bar{b}\bar{c}]} = +\frac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \\ \Sigma_{\bar{a}bc} = -\frac{1}{2}H_{\bar{a}bc}, \\ \Sigma_{a\bar{b}\bar{c}} = +\frac{1}{2}H_{a\bar{b}\bar{c}} \end{cases} \qquad \begin{cases} \Sigma_{\bar{a}\ b}^{\ \bar{a}} + \Sigma_{a\ b}^{\ a} = 0 - 4(\frac{1}{2}\partial_b\phi) = -2\partial_b\phi, \\ \Sigma_{a\ \bar{b}}^{\ a} + \Sigma_{\bar{a}\ \bar{b}}^{\ \bar{a}} = 0 - 2\partial_{\bar{b}}\phi. \end{cases}$$

We can always find a Σ to satisfy these conditions: $\Sigma_{a\ b}^{\ a} = \eta^{ac}\Sigma_{acb} = \eta^{ac}\Sigma_{(ac)b}$ only depends on the symmetric part of Σ_{abc}, therefore the two conditions are independent! We can always find a torsion-free compatible connection. These conditions however do not determine D uniquely.

In the aim of writing the components of D, we recall the form of $D^{\nabla(2)}$ in our frame

$$\begin{cases} D_a^{\nabla(2)} w_+^b = \nabla_a w_+^b \\ D_{\bar{a}}^{\nabla(2)} w_+^b = \nabla_{\bar{a}} w_+^b \\ D_a^{\nabla(2)} w_-^{\bar{b}} = \nabla_a w_-^{\bar{b}} \\ D_{\bar{a}}^{\nabla(2)} w_-^{\bar{b}} = \nabla_{\bar{a}} w_-^{\bar{b}}. \end{cases}$$

We can finally write the components of a generic torsion-free compatible connection:

$$\begin{cases} D_a w_+^b = \nabla_a w_+^b + \left(-\frac{1}{6}H_{a\ c}^{\ b}\right)^{(1)} w_+^c + \left(-\frac{2}{d-1}(\delta_a^b\partial_c\phi - \eta_{ac}\partial^b\phi)\right)^{(2)} w_+^c + \left(A_{a\ c}^{+b}\right)^{(3)} w_+^c, \\ D_{\bar{a}} w_+^b = \nabla_{\bar{a}} w_+^b + \left(-\frac{1}{2}H_{\bar{a}\ c}^{\ b}\right)^{(1)} w_+^c \\ D_a w_-^{\bar{b}} = \nabla_a w_-^{\bar{b}} + \left(\frac{1}{2}H_{a\ \bar{c}}^{\ \bar{b}}\right)^{(1)} w_-^{\bar{c}}, \\ D_{\bar{a}} w_-^{\bar{b}} = \nabla_{\bar{a}} w_-^{\bar{b}} + \left(\frac{1}{6}H_{\bar{a}\ \bar{c}}^{\ \bar{b}}\right)^{(1)} w_-^{\bar{c}} - \left(\frac{2}{d-1}(\delta_{\bar{a}}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi)\right)^{(2)} w_-^{\bar{c}} + \left(A_{\bar{a}\ \bar{c}}^{-\bar{b}}\right)^{(3)} w_-^{\bar{c}}, \end{cases}$$

where:

- We separated Σ into three parts: $(\cdot)^{(1)}$ which cancels the T_1 component of the D^∇ torsion, $(\cdot)^{(2)}$ which cancels the T_2 component, and $(\cdot)^{(3)}$ which does not contribute to the torsion.
- For all derivatives, the $(\cdot)^{(1)}$ part is equal to $(\cdot)^{(1)} = \eta^{bd}\Sigma_{[adc]}$. We used here the unbarred notation for simplicity's sake; the form of this equality is valid for all combinations of indices.
- For the derivatives with crossed indices (unbarred acting on barred or vice-versa), $\Sigma_{[a\bar{b}c]}$ and $\Sigma_{[a\bar{b}\bar{c}]}$ encapsulate all of $\Sigma_{a\bar{b}c}$ and $\Sigma_{a\bar{b}\bar{c}}$ respectively. Thus only the $(\cdot)^{(1)}$ terms appear in these two connections.
- For the derivatives with uncrossed indices (barred acting on barred, or unbarred acting on unbarred), the first two parts of $\Sigma_{a\ c}^{\ b}$ are explicitly $(\cdot)^{(1)} = \eta^{bd}\Sigma_{[adc]}$, $(\cdot)^{(2)} = t_{a\ c}^{\ b}$ any tensor such that the trace on its first two indices is $t_{a\ c}^{\ a} = -2\partial_b\phi$. In this paragraph, we used the unbarred notation for simplicity's sake; these equalities are also valid for three unbarred indices.

  We can verify that $t_{a\ c}^{\ b} = -\frac{2}{d-1}(\delta_a^b\partial_c\phi - \eta_{ac}\partial^b\phi)$ is valid for $(\cdot)^{(2)}$:

$$t_{a\ c}^{\ a} = -\frac{2}{d-1}(\delta_a^a\partial_c\phi - \eta_{ac}\partial^a\phi) = -\frac{2}{d-1}(d\partial_c\phi - \partial_c\phi) = -\frac{2}{d-1}(d-1)\partial_c\phi = -2\partial_c\phi.$$

  Note that in the supergravity context which we are interested in, we have $d = 10$, so $d-1 = 9$.
- The undetermined tensors $A^{+/-}$ satisfy

$$A^+_{abc} = -A^+_{acb}, \qquad \begin{cases} A^+_{[abc]} = 0, \\ A^{+a}_{a\ \ b} = 0, \end{cases}$$

$$A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}, \qquad \begin{cases} A^-_{[\bar{a}\bar{b}\bar{c}]} = 0, \\ A^{-\bar{a}}_{\bar{a}\ \ \bar{b}} = 0, \end{cases}$$

  so as to fit the connection compatibility criteria (left side), without contributing to the torsion (right side).

Remark. *The two cross derivatives are uniquely determined. The two others are not, however their contractions are:*

$$D_a w_+^a = \nabla_a w_+^a - \frac{1}{6}H_{a\ b}^{\ a} w_+^b - \frac{2}{d-1}(\delta_a^a\partial_b\phi - \eta_{ab}\partial^a\phi)w_+^b + A^{+a}_{a\ \ b}w_+^b = \nabla_a w_+^a - 2(\partial_a\phi)w_+^a,$$

where we use $H_{a\ b}^{\ a} = \eta^{ac}H_{acb} = \eta^{(ac)}H_{[acb]} = 0$ and $A^{+a}_{a\ \ b} = 0$. The same result follows for barred indices:

$$D_{\bar{a}}w_-^{\bar{a}} = \nabla_{\bar{a}}w_-^{\bar{a}} - 2(\partial_{\bar{a}}\phi)w_-^{\bar{a}}.$$

We will use these results in the second part of the following section.

4.3 Supergravity equations of motion and symmetry variations

4.3.1 Supersymmetry variations

The supersymmetry variations can be written in a locally $Spin(9,1) \times Spin(1,9)$ covariant form using the torsion-free compatible connection D.

The supersymmetry fermionic variations can be written in the simple forms

$$\begin{cases} \delta\psi^{+}_{\bar{a}} = D_{\bar{a}}\epsilon^{+} + \frac{1}{16}F_{\#}\gamma_{\bar{a}}\epsilon^{-}, \\ \delta\psi^{-}_{a} = D_{a}\epsilon^{-} + \frac{1}{16}F^{T}_{\#}\gamma_{a}\epsilon^{+}, \\ \delta\rho^{+} = \gamma^{a}D_{a}\epsilon^{+}, \\ \delta\rho^{-} = \gamma^{\bar{a}}D_{\bar{a}}\epsilon^{-}, \end{cases}$$

where $F_{\#} = \Lambda^{+}\not{F}^{(B,\phi)}(\Lambda^{-})^{-1}$, where $\Lambda^{\mp}$ are the $Spin(9,1)$ transformations corresponding to the Lorentz transformations $e^{\mp}_{a} = \Lambda^{\mp b}_{a}e_{a}$, and $\not{F}^{(B,\phi)} = \sum_{n}\frac{1}{n!}F^{(B,\phi)}_{a_1\ldots a_n}\gamma^{a_1\ldots a_n}$ with $F^{(B)} = e^{B_{(i)}}\wedge F_{(i)} = e^{B_{(i)}}\wedge\sum_{n}\mathrm{d}A^{(n-1)}_{(i)}$. $F_{(i)} = \mathrm{d}A_{(i)}$ is patched as $F_{(i)} = e^{\mathrm{d}\Lambda_{(ij)}}\wedge F_{(j)}$.

Remark. *As we assume the underlying manifold M possesses a spin structure, we can promote $O(9,1)\times O(1,9)$ to $Spin(9,1)\times Spin(1,9)$.*

For the bosonic fields, we have the variation of a generic $Spin(9,1)\times Spin(1,9)$ frame ($O(p,q)\times O(q,p)$ structure):

$$\begin{cases} \tilde{\delta}\hat{E}^{+}_{a} = (\delta\log\Phi)\hat{E}^{+}_{a} - (\delta\Lambda^{+}_{a\bar{b}})\hat{E}^{-\bar{b}}, \\ \tilde{\hat{E}}^{-}_{\bar{a}} = (\delta\log\Phi)\hat{E}^{-}_{\bar{a}} - (\delta\Lambda^{-}_{\bar{a}b})\hat{E}^{+b}, \end{cases}$$

where

$$\begin{cases} \delta\Lambda^{+}_{a\bar{a}} = \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \\ \delta\Lambda^{-}_{a\bar{a}} = \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \\ \delta\log\Phi = -2\delta\phi + \frac{1}{2}\delta\log(-g) = \bar{\epsilon}^{+}\rho^{+} + \bar{\epsilon}^{-}\rho^{-}. \end{cases}$$

The corresponding variations of the frames $\hat{e}^{\mp}$ are

$$\tilde{\delta}e^{+a}_{\mu} = \bar{\epsilon}^{+}\gamma_{\mu}\psi^{+a} + \bar{\epsilon}^{-}\gamma^{a}\psi^{-}_{\mu},$$

$$\tilde{\delta}e^{-\bar{a}}_{\mu} = \bar{\epsilon}^{+}\gamma^{\bar{a}}\psi^{+}_{\mu} + \bar{\epsilon}^{-}\gamma_{\mu}\psi^{-\bar{a}},$$

which both give the metric variation

$$\tilde{\delta}g_{\mu\nu} = 2\bar{\epsilon}^{+}\gamma_{(\mu}\psi^{+}_{\nu)} + 2\bar{\epsilon}^{-}\gamma_{(\mu}\psi^{-}_{\nu)},$$

as required. This can also be expressed in terms of the generalised metric G_{AB} as

$$\delta G_{a\bar{a}} = \delta G_{\bar{a}a} = 2(\bar{\epsilon^{+}}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}).$$

The variation of the RR potential A can be written as

$$\frac{1}{16}(\delta A_{\#}) = (\gamma^{a}\epsilon^{+}\bar{\psi}_{a}{}^{-} - \rho^{+}\bar{\epsilon}^{-}) \pm (\psi^{+}_{\bar{a}}\bar{\epsilon}^{-}\gamma^{\bar{a}} + \epsilon^{+}\bar{\rho}^{-},$$

where the upper sign corresponds to type IIA and the lower sign to type IIB.

4.3.2 Equations of motion

We will state how the paper finds we can rewrite the supergravity equations of motion with local $Spin(9,1)\times Spin(1,9)$ covariance using the following generalised notions of curvature:

- the generalised Ricci tensor, defined as $R_{a\bar{b}}w^{a}_{+} = [D_{a}, D_{\bar{b}}]w^{a}_{+}$ or equivalently $R_{\bar{a}b}w^{\bar{a}}_{+} = [D_{\bar{a}}, D_{b}]w^{\bar{a}}_{+}$.
- the generalised scalar curvature S such that $-\frac{1}{4}S\epsilon^{+} = (\gamma^{a}D_{a}\gamma^{b}D_{b} - D^{\bar{a}}D_{\bar{a}})\epsilon^{+}$, or alternatively $-\frac{1}{4}S\epsilon^{-} = (\gamma^{\bar{a}}D_{\bar{a}}\gamma^{\bar{b}}D_{\bar{b}} - D^{a}D_{a})\epsilon^{-}$.

We have the equations of motion

$$R_{a\bar{b}} + \frac{1}{16}\Phi^{-1}\left\langle F, \Gamma_{a\bar{b}}F\right\rangle = 0 \text{ for } g \text{ and } B,$$

$$S = 0 \text{ for } \phi,$$

$$\frac{1}{2}\Gamma^A D_A F = \mathrm{d}F = 0 \text{ for the RR fields.}$$

The bosonic pseudo-action is given by the simple expression

$$S_B = \frac{1}{2\kappa^2}\int (\Phi S + \frac{1}{4}\left\langle F, \Gamma^{(-)}F\right\rangle),$$

where Φ is a density.

The fermionic action takes the form

$$S_F = -\frac{1}{2\kappa^2}\int 2\Phi\big[\bar{\psi}^{+\bar{a}}\gamma^b D_b\psi^+_{\bar{a}} + \bar{\psi}^{-\bar{a}}\gamma^{\bar{b}} D_{\bar{b}}\psi^-_a + 2\bar{\rho}^+ D_{\bar{a}}\psi^{+\bar{a}} + 2\bar{\rho}^- D_a\psi^{-a}$$

$$-\bar{\rho}^+\gamma^a D_a\rho^+ - \bar{\rho}^-\gamma^{\bar{a}} D_{\bar{a}}\rho^- - \frac{1}{8}(\bar{\rho}^+ F_\#\rho^- + \bar{\psi}^+_{\bar{a}}\gamma^a F_\#\gamma^{\bar{a}}\psi^-_a)\big].$$

Varying this action with respect to the fermionic fields gives us the generalised geometry version of the fermionic equations of motion:

$$\begin{cases} \gamma^b D_b\psi^+_{\bar{a}} - D_{\bar{a}}\rho^+ = +\frac{1}{16}\gamma^b F_\#\gamma_{\bar{a}}\psi^-_b, \\ \gamma^{\bar{b}} D_{\bar{b}}\psi^-_a - D_a\rho^- = +\frac{1}{16}\gamma^{\bar{b}} F^T_\#\gamma_a\psi^+_{\bar{b}}, \\ \gamma^a D_a\rho^+ - D^{\bar{a}}\psi^+_{\bar{a}} = -\frac{1}{16}F_\#\rho^-, \\ \gamma^{\bar{a}} D_{\bar{a}}\rho^- - D^a\psi^-_a = -\frac{1}{16}F^T_\#\rho^+, \end{cases}$$

These supergravity equations are written in terms of torsion-free generalised connections, and therefore are manifestly covariant under local $Spin(9,1) \times Spin(1,9)$ transformations.

5 The Lichnerowicz bound

We will introduce the restriction of the eigenvalue spectrum of the Laplacian $\Delta f = \nabla^2 f = -\lambda f$ on an Einstein manifold, and try to find a similar result with our generalised tangent space, replacing ∇ with our generalised connection D, and contracting the connections with one of the generalised metrics we encountered.

5.1 Theorem for Einstein manifolds

Theorem. *For a compact Riemannian manifold*[8] (M, g) *of dimension* d *where* $R_{pm} = (d-1)g_{pm}$*, If* λ *is a non-zero eigenvalue of the positive Laplacian,* $-\nabla^2 f = \lambda f$*, then* $\lambda \geq d$*.*

[8]

Definition. *Riemannian manifold: Real, smooth manifold* M *equipped with a positive-definite inner product* g_p *on each fibre* T_pM *of the tangent bundle* TM*.*

Definition. *Einstein manifold: Riemannian or pseudo-Riemannian differentiable manifold whose Ricci tensor is proportional to the metric.*

Proof. To prove this inequality, we can use Bochner's identity:

$$\frac{1}{2}\nabla^p\nabla_p(\nabla_m f\nabla^m f) = (\nabla^m\nabla^p f)(\nabla_m\nabla_p f) - \nabla_m(\nabla^2 f)\nabla^m f + R_{mn}(\nabla^m f)(\nabla^n f).$$

We will name the left-hand side $H(f)$.

Remark. *We use a connection compatible with the contraction metric, so we can swap index heights in a contraction with a covariant derivative without worrying about a derivation of the metric.*

We have

$$H(f) = \frac{1}{2}\nabla^p\nabla_p(\nabla_m f\nabla^m f) = \nabla^p(\nabla_p(\nabla_m f)\nabla^m f) = (\nabla_p\nabla_m f)(\nabla^p\nabla^m f) + (\nabla^p\nabla_p\nabla_m f)\nabla^m f,$$

where

$$\nabla^p\nabla_p\nabla_m f = \nabla^p\nabla_m\nabla_p f = ([\nabla_p,\nabla_m] + \nabla_m\nabla_p)\nabla^p f = (R_{pm\ \ n}^{\ \ \ p}\nabla^n + \nabla_m\nabla_p\nabla^p)f$$

$$= (R_{pm\ \ n}^{\ \ \ p}\nabla^n - \lambda\nabla_m)f = (R_{mn}\nabla^n - \lambda\nabla_m)f = ((d-1)g_{pm}\nabla^p - \lambda\nabla_m)\,f = (d-1-\lambda)\nabla_m f,$$

where use the fact that we are in coordinate indices, so $R_{pm\ \ n}^{\ \ \ p}v^n = [\nabla_p,\nabla_m]v^p$. Therefore we have

$$H(f) = (\nabla_p\nabla_m f)(\nabla^p\nabla^m f) + (d-1-\lambda)\nabla_m\nabla^m f.$$

We can separate $\nabla_m\nabla_n f$ into a traceless part and a part containing its trace. Its trace is equal to

$$g^{mn}\nabla_m\nabla_n f = \nabla^m\nabla_m f = -\lambda f,$$

the same as

$$g^{nm}(-\frac{\lambda}{n}g_{mn}f) = -\lambda f.$$

By defining Δ_{pm} such that

$$\nabla_p\nabla_m f = -\frac{\lambda}{n}g_{pm}f + \Delta_{pm},$$

then Δ_{pm} is traceless, $g^{pm}\Delta_{pm} = 0$, and consequently

$$(\nabla_p\nabla_m f)(\nabla^p\nabla^m f) = \frac{\lambda^2 f^2}{d} + \Delta_{pm}\Delta^{pm}.$$

Integrating $H(f) = \frac{1}{2}\nabla^p\nabla_p(\nabla_m f\nabla^m f)$ over spacetime, we have

$$\int\sqrt{-g}H(f) = 0,$$

seeing as H is a total derivative; we are using the assumption that all quantities tend to zero on the edges of spacetime. This integral is equal to

$$\int\sqrt{-g}H(f) = \int\sqrt{-g}\left[(\nabla_m\nabla_n f)(\nabla^m\nabla^n f) + (d-1-\lambda)\nabla_m f\nabla^m f\right]$$

$$= \int\sqrt{-g}(\frac{\lambda^2 f^2}{d} + \Delta_{mn}\Delta^{mn}) + \int\sqrt{-g}(d-1-\lambda)\nabla_m f\nabla^m f$$

By integrating by parts $\frac{\lambda}{d}(\nabla_m f\nabla^m f)$, we obtain

$$\int\sqrt{-g}\frac{\lambda}{d}(\nabla_m f\nabla^m f) = \int\sqrt{-g}\frac{\lambda}{d}\nabla_m(f(\nabla^m f)) - \int\sqrt{-g}\frac{\lambda}{d}f(-\lambda f) = \int\sqrt{-g}\frac{\lambda^2 f^2}{d},$$

where the left term is null because of the total derivative. We recognise the first term in our integration of $H(f)$; we can write

$$\int \sqrt{-g}H(f) = 0 = \int \sqrt{-g}(d-1-\lambda+\frac{\lambda}{d})(\nabla_m f)(\nabla^m f) + \int \sqrt{-g}\Delta_{mn}\Delta^{mn},$$

Since the metric is positive-definite, the second integral is positive, and $(\nabla_m f)(\nabla^m f)$ also, which implies

$$\begin{aligned}
&(d-1-\lambda+\frac{\lambda}{d}) \leq 0 \\
\iff\ & \lambda(1-\frac{1}{d}) \geq d-1 \\
\iff\ & \lambda \geq \frac{d-1}{1-\frac{1}{d}} = \frac{d^2-d}{d-1} = d,
\end{aligned}$$

which marks the end of the proof. □

Modelling this build-up to the Lichnerowicz eigenvalue spectrum on an Einstein manifold, can we find a similar result replacing the tangent space with the generalised tangent space?

5.2 Lichnerowicz in generalised geometry?

In analogy to the setting of the Ricci tensor to be proportional to the metric on the tangent space of an Einstein manifold, we will set $R_{MN} = G_{MN}$ in our generalised tangent space.

Choosing the Ricci tensor to be proportional to the metric G implies, in the $\{\hat{E}^+_a\}\cup\{\hat{E}^-_{\bar{a}}\}$ frame,

$$R_{AB} = \begin{pmatrix} R_{ab} & R_{a\bar{b}} \\ R_{\bar{a}b} & R_{\bar{a}\bar{b}} \end{pmatrix} = \kappa G_{AB} = \kappa \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix},$$

so the Ricci tensor must satisfy

$$\begin{cases} R_{ab} = \kappa\eta_{ab}, \quad R_{\bar{a}\bar{b}} = \kappa\eta_{\bar{a}\bar{b}}, \\ R_{a\bar{b}} = R_{\bar{a}b} = 0. \end{cases}$$

We want to see if we can use an identity similar to Bochner's on an Einstein manifold, to find a similar restriction on the eigenvalue spectrum in the Laplacian equation

$$-D^M D_M f = \lambda f.$$

There are several equations of this form we could try to solve, since we have two choices of metric we could contract with, one of which can be decomposed into two separate metrics: we could contract two $O(p,q)\times O(q,p)$ - covariant derivatives with the generalised metric η^{AB} or the additional structure metric G^{AB}; we could also contract two $O(p,q)$-covariant derivatives D^+ or D^- with the metric g. We will focus here on the first two. We have

$$G^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta^{\bar{a}\bar{b}} \end{pmatrix}, \quad \eta^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & -\eta^{\bar{a}\bar{b}} \end{pmatrix}.$$

and the following two possible equations:

$$\begin{cases} \eta^{MN}D_N D_M f = -\lambda f \text{ contracted with } \eta, \\ G^{MN}D_N D_M f = -\lambda f \text{ contracted with } G. \end{cases}$$

We will name these cases 1 and 2 respectively.

$$D^M D_M f = D^A D_A f = (\eta^{ab} D_a^+ D_b^+ \pm \eta^{\bar{a}\bar{b}} D_{\bar{a}}^+ D_{\bar{b}}^+) f$$
$$= D_a^+ (\partial^a f) \pm D_{\bar{a}}^- (\partial^{\bar{a}} f) = \nabla_a (\partial^a f) - 2(\partial_a \phi)(\partial^a f) \pm \nabla_{\bar{a}} (\partial^{\bar{a}} f) \mp 2(\partial_{\bar{a}} \phi)(\partial^{\bar{a}} f)$$
$$= \nabla_a (\partial^a f) - 2(\partial_a \phi)(\partial^a f) \pm (\nabla_a (\partial^a f) - 2(\partial_a \phi)(\partial^a f)),$$

where the lower sign corresponds to case 1 (η^{AB}), the upper sign to case 2 (G^{AB}), and where both barred and unbarred indices are contracted with the same metric g. We can see that if we use η^{AB} to contract the connections, we have $D^M D_M f = 0$. We can say that the spectrum of the eigenvalue λ is $\{0\}$, but this is not very interesting. We can discard case 0.

If we use G^{AB} (case 1), we obtain

$$G^{AB} D_A D_B f = 2\nabla_a (\partial^a f) - 4(\partial_a \phi)(\partial^a f).$$

To obtain Bochner's identity on an Einstein manifold, we use the tensorial curvature equality $[\nabla_p, \nabla_m] v^p = R_{pm\ \ n}^{\ \ \ p} v^n$ in coordinate indices. However in generalised geometry the curvature $R(U, V, W)$ is not always a tensor; we recall this is the case only for $\langle U, V \rangle = 0$. By following Bochner's steps, we will uncover conditions that might need to be imposed for a tensorial measure of the curvature to appear in our equality.

We write

$$\frac{1}{2} D_B D^B (D_A f D^A f) = D_B ((D^B D_A f) D^A f) = (D^2 D_A f) D^A f + (D_A D_B f)(D^A f D^B f).$$

It is from $D^2 D_A f$ that we would normally obtain our R^{AB} Ricci tensor:

$$D^2 D_A f = D_B D_A (D^B f) = ([D_B, D_A] + D_A D_B) D^B f,$$

where we use $[D_B, D_A] f = 0$. This expression only makes sense if $[D_B, D_A]$ is a tensor, which is equivalent to the curvature $R(\hat{E}_B, \hat{E}_A, W)$ being a tensor. Therefore, for all indices we are summing over, we need $\left\langle \hat{E}_B, \hat{E}_A \right\rangle = 0$, where $\langle , \rangle$ represents an η^{AB} contraction. This is verified for $A \leq d,\ B > d$, or vice-versa, as we can see from the block-diagonal form of the η^{AB} matrix.

If we sum only over $\{A \leq d, B > d\} \cup \{A > d, B \leq d\}$, we have $([D_B, D_A] D^B f) D^A f = R^B_{CBA} D^C f D^A f$, which is then equal to $R_{c\bar{a}} D^c f D^{\bar{a}} f + R_{\bar{c}a} D^{\bar{c}} f D^a f$. However this is zero due to the condition of the Ricci tensor being proportional to the metric G.

For a tensorial measure of curvature to appear in this equation, which we hope to use to impose a boundary on the eigenvalue spectrum, we would need some combination of derivatives to form the scalar curvature S and for this to be non-zero. This is not the case of full type II supergravity. However we could consider compactifications of the ten-dimensional theory of the form:

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2(\mathbb{R}^{9-d,1}) + \mathrm{d}s^2_d$$

where the first term is a flat metric on $\mathbb{R}^{9-d,1}$ and the second term a general metric on the d-dimensional internal manifold. This would be equivalent to adding a term of the form $e^{-2\phi} C$ in the supergravity action, similar to a cosmological constant.

The field equations of motions on the internal space would then have the same generalised geometry form, but with the structure $O(p, q) \times O(q, p) = O(d) \times O(d) \subset O(d, d) \times \mathbb{R}^+$. One crucial difference would be in the dilaton equation of motion, which would be changed from $S = 0$ to $S = C \neq 0$, some non-zero constant dependent on the constant curvature of the flat space.

The motivation for this is the following. An eigenvalue λ for the restriction of D^2 to the internal space translates a wave equation $D^2 f = 0$ on the ten-dimensional space, into a four-dimensional equation on the flat space with mass term λ. Thus, if we find in analogy to the Lichnerowicz bound, a bound for the eigenvalue spectrum on the restriction of D^2 to the internal space, this would give restrictions on the mass term in flat space, which has a physical meaning, as the flat space is observable!

6 Conclusion

Following the generalised geometrical constructions in the *Supergravity as Generalised Geometry I* paper, we defined a generalised tangent space that geometrises the NSNS bosonic sector, incorporating the patching and symmetry algebra of the potential B into the generalised geometry. We were able to construct an algebra (Courant bracket) of generalised Lie derivatives (Dorfman derivatives), combining the usual symmetry algebra of diffeomorphisms with the B-field gauge transformations.

One way of seeing this is with a globally-defined split frame, from which we can deduce the isomorphism $E \simeq TM \oplus T^*M, V \mapsto V^{(B)}$. This enables us to translate a conventional, diffeomorphism-reflecting connection ∇ acting on components of sections of $TM \oplus T^*M$, to a corresponding generalised connection $D^{\nabla(1)}$ on E, which encodes these conventional symmetry operations while being compatible with the structure of E described in the next paragraph. The patching of B lies in the split frame itself, and a B-field gauge transformation marks the passage from one split frame in E to another.

We found that the definition of the generalised tangent space is consistent with an $O(d,d)$ metric which admits an $O(p,q) \times O(q,p)$ sub-structure; this is crucial to be able to relate this space to type II supergravity.

Indeed, regarding the NSNS bosonic fields, we have shown that an $O(9,1) \times O(1,9) \subset O(10,10) \times \mathbb{R}^+$ generalised structure is parametrised by a metric g of signature $(9,1)$, a two-form B patched as in supergravity, and a dilaton ϕ: At each point $x \in M$, $\{g, B, \phi\} \in \frac{O(10,10)\times\mathbb{R}^+}{O(9,1)\times O(1,9)}$. Thus this substructure captures the NSNS bosonic fields, which are packaged into the generalised metric and conformal factor (G, ϕ). We note that we wrote a generic $O(p,q) \times O(q,p)$ structure explicitly; it resembles the split frame for the corresponding B but is further conformal, thus encoding the dilaton ϕ and living in $\tilde{E}$, while encoding a metric g on the tangent space through the added one-form terms factored by the metric coefficients. Thus these frames encode all of the NSNS bosonic fields and the symmetries of ϕ and B, and a generalised connection $D^{\nabla(2)}$ corresponding to conventional connections on the $TM \oplus TM$ components in this frame, enables us to compatibly express the conventional diffeomorphism symmetries.

Regarding the other type II supergravity fields, as we assume the underlying manifold possesses a spin structure, we are able to promote $O(10,10)$ to $Spin(10,10)$ and the subgroup to $Spin(9,1) \times Spin(1,9)$; the RR fields transform as $Spin(10,10)$ spinors, and the type II fermionic degrees of freedom are spinor and vector-spinor representations of $Spin(9,1) \times Spin(1,9)$. In the final results of the paper, the bosonic and fermionic actions, leading equations of motion and supersymmetry variations are all rewritten in generalised, simple, $Spin(9,1) \times Spin(1,9)$-covariant form using the torsion-free compatible connection D we arrive at in this work.

Defining the torsion of a generalised connection in analogy to the conventional definition, the nature of D^{∇} enables us to explicitly calculate the generalised torsion for a connection corresponding to a torsion-free ∇. Having these explicit calculations, we were able to deduce the form of a torsion-free generalised connection, as a generic generalised connection can always be written $D_M W^A = D^{\nabla}_M W^A + \Sigma_M{}^A{}_B W^B$. In the substructure $O(p,q) \times O(q,p) \subset O(d,d) \times \mathbb{R}^+$, i.e. with the connection in the adjoint action of this group and acting within the corresponding sub-bundle, a compatible torsion-free connection D corresponds to what we call the generalised analogue of the Levi-Civita connection. As the $O(p,q) \times O(q,p)$ sub-bundle is a direct sum $C_+ \oplus C_-$, we find that D is divided into four $O(p,q) \times O(q,p)$ covariant operators, two of which are not unique. Though this could pose a problem for applications to supergravity, which is what we are ultimately interested in, we can form unique operators for instance from contracting their indices.

In analogy to the conventional construction, we were also able to define a generalised curvature $R(U, V, W)$, though we found this is not always tensorial: We need $\langle U, V \rangle = 0$. This requires the generalised Ricci tensor and scalar curvature to be defined differently than conventionally. This divergence from the conventional curvature was encountered as a problem when we explored the Lichnerowicz problem in generalised geometry: where we needed the curvature to be tensorial and contract into the Ricci tensor, it could not. An avenue to explore in this particular problem but also more generally to avoid this issue, would be to find combinations and contractions of generalised connection operators that form the scalar curvature, which we would want to be non-zero. As this measure of generalised curvature is null in type II supergravity, which we can see in the equation of motion for the dilaton, our analysis of the problem would need to be reduced to the internal space of a compactification of the full ten-dimensional supergravity theory, where the metric is decomposed into a flat metric on $\mathbb{R}^{9-d,1}$ and a general internal metric on the d-dimensional internal manifold. Such an analysis could still lead to interesting physical results, as an eigenvalue on the internal space equation would correspond to a mass in the physical flat space.

7 References

Base article for this work:

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